Nonlinear Differential Equations

and The Beauty of Chaos

Examples of nonlinear equations

Simple harmonic oscillator (linear ODE)

$$m\frac{d^2x(t)}{dt} = -kx(t)$$

More complicated motion (nonlinear ODE)

$$m\frac{d^2x(t)}{dt} = -kx(t)(1-\alpha x(t))$$

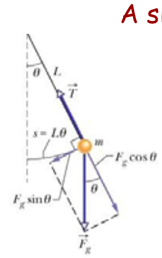
Other examples: weather patters, the turbulent motion of fluids

Most natural phenomena are essentially nonlinear.

What is special about nonlinear ODE?

- ⇨ For solving nonlinear ODE we can use the same methods we use for solving linear differential equations
- ⇨ What is the difference?
- ⇨ Solutions of nonlinear ODE may be simple, complicated, or chaotic
- ⇨ Nonlinear ODE is a tool to study nonlinear dynamic: chaos, fractals, solitons, attractors

A simple pendulum



Model: 3 forces

- gravitational force
- frictional force is proportional to velocity
- periodic external force

$$I\frac{d^2\theta}{dt^2} = \tau_g + \tau_f + \tau_{ext}$$

$$\tau_g = -mgL\sin(\theta), \quad \tau_f = -\beta\frac{d\theta}{dt},. \quad \tau_{ext} = F\cos(\omega t)$$

Equations

$$\frac{d^2\theta}{dt^2} = -\omega_0^2\sin(\theta) - \alpha\frac{d\theta}{dt} + f\cos(\omega t)$$

$$\omega_0^2 = \frac{mgL}{I} = \frac{g}{L}, \quad \alpha = \frac{\beta}{mL^2}, \quad f = \frac{F}{mL^2}$$

Computer simulation: there are very many web sites with Java animation for the simple pendulum

Case 1: A very simple pendulum

$$\frac{d^2\theta}{dt^2} = -\omega_0^2\sin(\theta)$$

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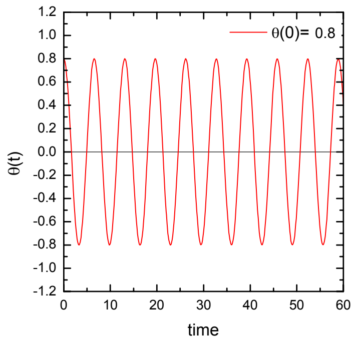

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Is there any difference between the nonlinear pendulum

$$\boxed{\frac{d^2\theta}{dt^2} = -\omega_0^2 \sin(\theta)}$$

and the linear pendulum?

$$\boxed{\frac{d^2\theta}{dt^2} = -\omega_0^2 \theta}$$

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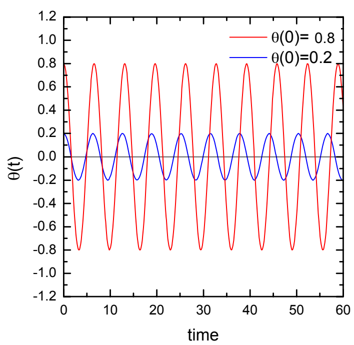

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Amplitude dependence of frequency

- For small oscillations the solution for the nonlinear pendulum is periodic with

$$\omega = \omega_0 = \sqrt{g/L}$$

- For large oscillations the solution is still periodic but with frequency

$$\omega < \omega_0 = \sqrt{g/L}$$

- explanation:

$$\sin(\theta) \approx \theta - \frac{1}{2}\theta^2 + \ldots$$

$$\sin(\vartheta) < \theta$$

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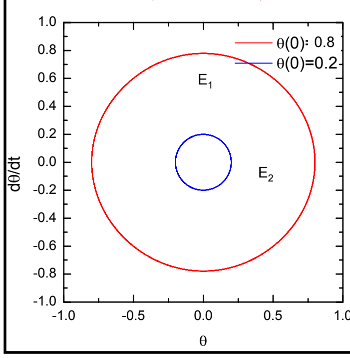
Phase-Space Plot
velocity versus position



phase-space plot is a very good way to explore the dynamic of oscillations

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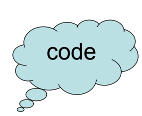
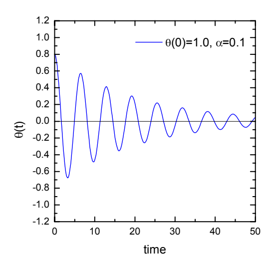
Case 2: The pendulum with dissipation

$$\boxed{\frac{d^2\theta}{dt^2} = -\omega_0^2 \sin(\theta) - \alpha \frac{d\theta}{dt}}$$





How about frequency in this case?

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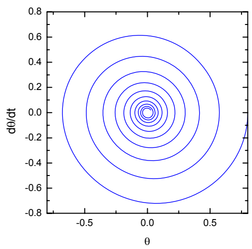
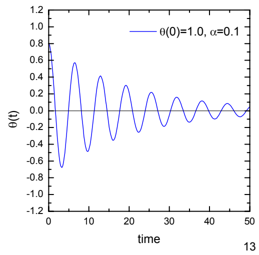
Phase-space plot for the pendulum with dissipation





Case 3: Resonance and beats

$$\frac{d^2\theta}{dt^2} = -\omega_0^2 \sin(\theta) + f\cos(\omega t)$$

- When the magnitude of the force is very large – the system is overwhelmed by the driven force (*mode locking*) and the are no beats
- When the magnitude of the force is comparable with the magnitude of the natural restoring force the beats may occur

code

Beats

- In beating, the natural response and the driven response add:

$$\theta \approx \theta_0 \sin(\omega t) + \theta_0 \sin(\omega_0 t) = 2\theta_0 \cos\left(\frac{\omega-\omega_0}{2}t\right)\sin\left(\frac{\omega+\omega_0}{2}t\right)$$

mass is oscillating at the average frequency $(\omega+\omega_0)/2$ and an amplitude is varying at the slow frequency $(\omega-\omega_0)/2$

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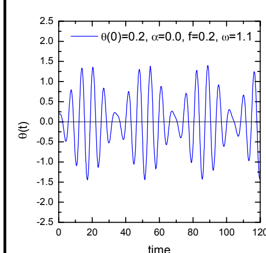
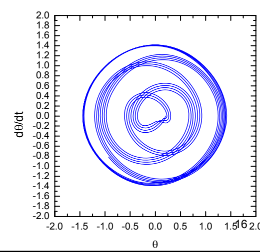
Example: beats





Resonance

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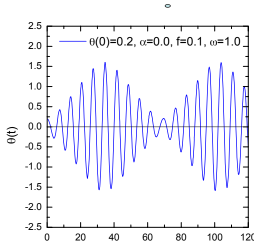
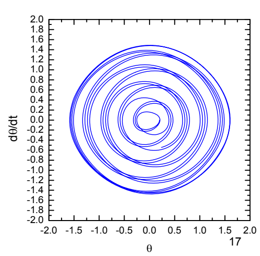


For a simple harmonic oscillator the amplitude of oscillations increases without bound



That is not true for the nonlinear oscillator

Case 4: Complex Motion

$$\frac{d^2\theta}{dt^2} = -\omega_0^2 \sin(\theta) - \alpha\frac{d\theta}{dt} + f\cos(\omega t)$$

- We have to compare the relative magnitude of the natural restoring force, the driven force and the frictional force
- The most complex motion one would expect when the three forces are comparable

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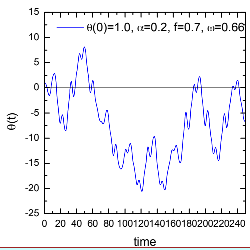
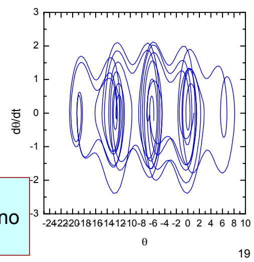
Case 4: Chaotic Motion

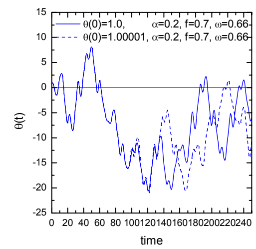


Chaotic motion is not random!



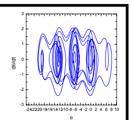
Chaos is the deterministic behavior of a system displaying no discernable regularity

Case 4: Chaotic Motion



- A chaotic system is one with an extremely high sensitivity to parameters or initial conditions
- The sensitivity to even miniscule changes is so high that, in practice, it is impossible to predict the long range behavior unless the parameters are known to infinite precision (which they never are in practice)

Measuring Chaos



How do we know if a system is chaotic?

The most important characteristic of chaos is sensitivity to initial conditions.

Sensitivity to initial conditions implies that our ability to make numerical predictions of its trajectory is limited.

How can we quantify this lack of predictably?

This divergence of the trajectories can be described by the Lyapunov exponent λ, which is defined by the relation:

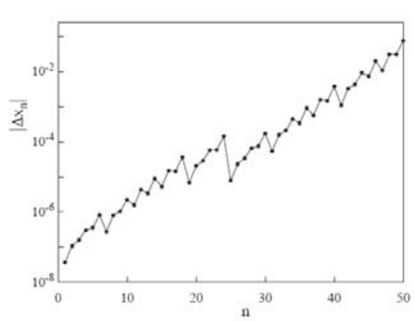
$$|\Delta x_n| = |\Delta x_0| e^{\lambda_n}$$

where Δxn is the difference between the trajectories at time n.

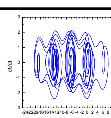
If the Lyapunov exponent λ is positive, then nearby trajectories diverge exponentially.

Chaotic behavior is characterized by the exponential divergence of nearby trajectories.

$$|\Delta x_n| = |\Delta x_0| e^{\lambda_n}$$



Chaotic structure in phase space



1. Limit cycles: ellipse-like figures with frequencies greater then ω_0
2. Strange attractors: well-defined, yet complicated semi-periodic behavior. Those are highly sensitive to initial conditions. Even after millions of observations, the motion remains *attracted* to those paths
3. Predictable attractors: well-defined, yet fairy simple periodic behaviors that not particularly sensitive to initial conditions
4. Chaotic paths: regions of phase space that appear as filled-in bands rather then lines

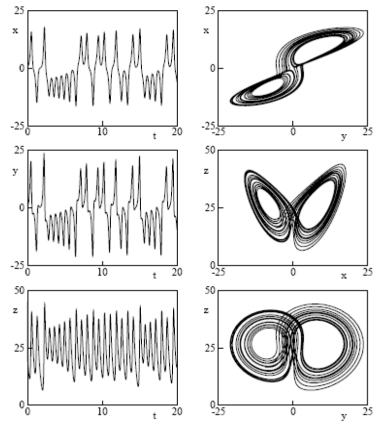
The Lorenz Model & the *butterfly effect*

- In 1962 Lorenz was looking for a simple model for weather predictions and simplified the heat-transport equations to the three equations

$$\frac{dx}{dt} = 10(y-x)$$
$$\frac{dy}{dt} = -xz+28x-y$$
$$\frac{dz}{dt} = xy-\frac{8}{3}z$$

- The solution of these simple nonlinear equations gave the complicated behavior that has led to the modern interest in chaos

Example



Hamiltonian Chaos

The Hamiltonian for a particle in a potential

$$H = \frac{1}{2m}(p_x^2+p_y^2+p_z^2)+V(x,y,z)$$

for N particles – 3N degrees of freedom

Examples: the solar system, particles in EM fields, ...
more specific example: the rings of Saturn

Attention: no dissipation!

Constants of motion: Energy, Momentum (linear, angular)

When a number of degrees of freedom becomes large, the possibility of chaotic behavior becomes more likely.

Summary

- The simple systems can exhibit complex behavior
- Chaotic systems exhibit extreme sensitivity to initial conditions.

Practice

- Duffing Oscillator

$$\frac{d^2x}{dt^2}+\alpha\frac{dx}{dt}-\frac{1}{2}x(1-x^2) = f\cos(\omega t)$$

- Write a program to solve the Duffing model. Is there a parametric region in (α, f, ω) where the system is chaotic

Fourier Analysis of Nonlinear Oscillations

- The traditional tool for decomposing both periodic and non-periodic motions into an infinite number of harmonic functions
- It has the distinguishing characteristic of generating a periodic approximations

Fourier series

For a <u>periodic</u> function

$$y(t+T)=y(t)$$

one may write

$$y(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos(n\omega t)+b_n\sin(n\omega t)\right),\qquad \omega=\frac{2\pi}{T}$$

The Fourier series is a "best fit" in the least square sense of data fitting

A general function may contain infinite number of components. In practice a good approximation is possible with about 10 harmonics

Coefficients:

the coefficients are determined by the standard technique for orthogonal function expansion

$$a_n=\frac{2}{T}\int_0^T\cos(n\omega t)y(t)dt,$$

$$b_n=\frac{2}{T}\int_0^T\sin(n\omega t)y(t)dt,$$

$$\omega=\frac{2\pi}{T}$$

Fourier transform

The right tool for non-periodic functions

$$y(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}Y(\omega)e^{i\omega t}d\omega$$

and the inverse transform is

$$Y(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}y(t)e^{-i\omega t}dt$$

a plot of $|Y(\omega)|^2$ versus ω is called the power spectrum

Spectral function

If $y(t)$ represent the response of some system as a function of time, $Y(\omega)$ is a spectral function that measures the amount of frequency ω making up this response

Methods to calculate Fourier transform

- Analytically
- Direct numerical integration
- Discrete Fourier transform
  (for functions that are known only for a finite number of times t_k
- Fast Fourier transform (FFT)

Discrete Fourier transform

- Assume that a function y(t) is sampled at a discrete number of N+1 points, and these times are evenly spaced
- Let T is the time period for the sampling:
  a function y(t) is periodic with T, y(t+T)=y(t)
- The largest frequency for this time interval is
  $\omega_1=2\pi/T$ and $\omega_n=n\omega_1=n2\pi/T=n2\pi/(Nh)$

Discrete Fourier transform

- The discrete Fourier transform, after applying a trapezoid rule

$$Y(\omega_n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega_n t} y(t) dt = \frac{h}{\sqrt{2\pi}} \sum_{k=1}^{N} e^{-i\frac{2\pi k n}{N}} y_k$$

$$y(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega_n t} Y(\omega) d\omega = \frac{\sqrt{2\pi}}{hN} \sum_{n=1}^{N} e^{i\frac{2\pi n t}{hN}} Y(\omega_n)$$

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DFT in terms of separate real and imaginary parts

$$e^{ix} = \cos(x) + i\sin(x)$$

$$\begin{aligned} Y(\omega_n) &= \frac{h}{\sqrt{2\pi}} \sum_{k=1}^{N} [(\cos(2\pi kn/N)\,\mathrm{Re}(y_k) \\ &+ \sin(2\pi kn/N)\,\mathrm{Im}(y_k)) \\ &+ i(\cos(2\pi kn/N)\,\mathrm{Im}(y_k) \\ &- \sin(2\pi kn/N)\,\mathrm{Re}(y_k))] \end{aligned}$$

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Practice for the simple pendulum

Solve the simple pendulum for harmonic motion, beats, and chaotic motion (the dissipation and driven forces are included)

- Decompose your numerical solutions into a Fourier series. Evaluate contribution from the first 10 terms
- Evaluate the power spectrum from your numerical solutions

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