Chapter 10

Dynamics of Rotational Motion

Rotational dynamics

Define physical quantity that is the angular analog of force: something that produces change in angular velocity

Torque:
determines effectiveness of force to produce rotation

Definition of the torque:

$$\tau_O = \vec{r}_O \times \vec{F}$$
Torque

Calculating torque:
\[ \tau_O = r_O F \sin \theta = (r_O \perp F) = r_O F \perp. \]

Lever arm
\[ (r_O \perp) = r_O \sin \theta \]

Units: (N·m) ("Newton-meter")

Distinguish from joule = N·m:
For torque, effective parts of force and distance are perpendicular.
For work, effective parts of force and distance are parallel.

Torque and dynamics of rotation

Equation of motion for single particle
\[ \vec{F} = \frac{d\vec{p}}{dt} \Rightarrow \vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt}. \]

\[ \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times m\vec{v} + \vec{F} \times \frac{d\vec{p}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt}. \]

Torque equation
\[ \tau = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{L}}{dt}. \]

Angular momentum
\[ \vec{L} \equiv \vec{r} \times \vec{p} \quad , \quad |\vec{L}| = rp \sin \theta. \]

Zero torque: \( \vec{L} \) constant
Angular momentum of point particle is conserved if net torque on it vanishes.
Conservation of angular momentum for a system of particles

- Total angular momentum of the system:
  \[ \vec{L} = \vec{L}_1 + \vec{L}_2 + \ldots + \vec{L}_N = \sum_{i=1}^{N} \vec{L}_i. \]

- Total torque on the system:
  \[ \vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2 + \ldots + \vec{\tau}_N = \sum_{i=1}^{N} \vec{\tau}_i. \]

- Torque produced by external forces and internal forces:
  \[ \vec{\tau}_i = \vec{\tau}_i^{\text{ext}} + \sum_{j=1}^{N} \vec{\tau}_{ji}. \]

- Internal torques cancel because of 3rd Law:
  \[ \frac{d\vec{L}}{dt} = \vec{\tau}^{\text{ext}}. \]

- Total angular momentum of a system of particles is conserved if the external torque on the system is zero.

Angular momentum for simple rotation

System of particles (or solid body) rotating with angular velocity \( \vec{\omega} \) about a fixed axis: \( \vec{\tau}_i = \vec{\omega} \times \vec{r}_i. \)

\[ \vec{L} = \sum_{i=1}^{n} \vec{L}_i = \sum_{i=1}^{n} \vec{r}_i \times m_i \vec{v}_i = \sum_{i=1}^{n} m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) \]

Double vector product

\[ \vec{r} \times (\vec{\omega} \times \vec{r}) \]

Take \( \vec{r} \perp \vec{\omega} \):

\( \vec{r} = \vec{r}_\perp \)

\[ \vec{\omega} \times \vec{r}_\perp = \omega r_\perp \hat{k} \]

\[ \Rightarrow \vec{\omega} \times \vec{r}_\perp = \omega r_\perp \hat{k} \]

\[ \Rightarrow \vec{r}_\perp \times \omega r_\perp \hat{k} = \omega r_\perp ^2 \hat{i} \]

\[ \Rightarrow \vec{r}_\perp \times (\vec{\omega} \times \vec{r}_\perp) = \vec{\omega} r_\perp ^2 \]

General case:

\[ \vec{r} = \vec{r}_\perp + \vec{\omega} \frac{\vec{\omega} \times \vec{r}}{\vec{\omega} \cdot \vec{r}} \]

Note:

\[ \vec{\omega} \times \vec{\omega} = 0 \Rightarrow \vec{\omega} \times \vec{r} = \vec{\omega} \times \vec{r}_\perp \]

Extra term in second product:

\[ \frac{\partial}{\partial \vec{r}} \vec{\omega} \times (\vec{\omega} \times \vec{r}_\perp) \]

\[ \vec{\omega} \times (\vec{\omega} \times \vec{r}_\perp) = -\vec{r}_\perp \omega^2 \]

\[ \Rightarrow \vec{r} \times (\vec{\omega} \times \vec{r}) = \omega r_\perp ^2 - \vec{r}_\perp (\vec{\omega} \cdot \vec{r}) \]
Angular momentum and moment of inertia

Angular momentum of rotating body:

\[ \vec{L} = \sum_{i=1}^{n} m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) = \sum_{i=1}^{n} m_i [\vec{\omega} \times (\vec{r}_i \cdot \vec{r}_i)] \]

\[ r_{i\perp} = \text{distance from particle to axis of rotation} \]

\[ \vec{\omega} \cdot \vec{r} \equiv \omega x \] characterizes its position in direction parallel to axis of rotation

If system is symmetric with respect to axis of rotation:
for each particle at \((x, r_{i\perp})\), there is particle at \((x, -r_{i\perp})\)
\[ \Rightarrow \text{terms } r_{i\perp} (\vec{\omega} \cdot \vec{r}_i) \text{ cancel in the sum} \]

Simple result:

\[ \vec{L} |_{\text{Symm}} = \vec{\omega} \sum_{i=1}^{n} m_i r_{i\perp}^2 = I_0 \vec{\omega} \]

Torque and angular acceleration for rigid body

Net force acting on particle:
radial component \(F_{i\text{rad}}\)
tangential component \(F_{i\text{tan}}\)

From 2nd Law:

\[ F_{i\text{tan}} = m_i a_{i\text{tan}} \]

In terms of angular acceleration:
\(a_{i\text{tan}} = r_i \alpha_z\)

\[ r_i F_{i\text{tan}} = m_i r_i^2 \alpha_z \]
\[ \Rightarrow \tau_{iz} = m_i r_i^2 \alpha_z = I_{iz} \alpha_z. \]

Summing over particles:

\[ \sum_i \tau_{iz} = \left( \sum_i m_i r_i^2 \right) \alpha_z = I_z \alpha_z \]

Analog of 2nd Law:

\[ \sum \tau_z = I_z \alpha_z \]
A falling block

A uniform disk with mass $M=2.5$ kg, and radius $R=20$ cm mounted on a horizontal axis. A block with mass $m=1.2$ kg hangs from a massless cord that is wrapped around the rim of the disk. Find the acceleration of the falling block, the angular acceleration, and the tension in the cord.

\[
T - mg = ma
\]
\[
\tau_{net} = 1\alpha
\]
\[
-RT = \frac{1}{2} MR^2 \alpha
\]
\[
a = \alpha R
\]
\[
a = -g \frac{2m}{M + 2m} = -4.8 \text{ m/s}^2
\]
\[
T = \frac{1}{2} Ma
\]
\[
T = 6.0 \text{ N}
\]
\[
\alpha = \frac{a}{R} = -24 \text{ rad/s}^2
\]

Work done by torque

Definition of torque: $\vec{\tau} = \vec{r} \times \vec{F}$

If $\vec{F}$ is tangential force: $\vec{r} \perp \vec{F}$ and $\tau = rF \Rightarrow \vec{F} = \tau/r$

Elementary work:

\[
dW = \vec{F} \cdot d\vec{s} = Frd\theta = \frac{\tau}{r} rd\theta \Rightarrow dW = \tau d\theta
\]

Work due to torque:

\[
\Rightarrow W = \int \tau \cdot d\theta
\]

Rotational energy: $K = I\omega^2/2$

$\Rightarrow$ work-energy theorem $W = \Delta K$ gives

\[
W = \frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_i^2
\]
Energy for combined translational and rotational motion

Velocity of $i$th particle
$$\vec{v}_i = \vec{v}_{cm} + \vec{v}_i^r$$

Kinetic energy of $i$th particle
$$K_i = m_i\vec{v}_i^2 / 2 = m_i(\vec{v}_{cm} + \vec{v}_i^r) \cdot (\vec{v}_{cm} + \vec{v}_i^r)$$
$$= \frac{1}{2} m_i (\vec{v}_{cm} \cdot \vec{v}_{cm} + 2 \vec{v}_{cm} \cdot \vec{v}_i^r + \vec{v}_i^r \cdot \vec{v}_i^r)$$
$$= \frac{1}{2} m_i (v_{cm}^2 + 2\vec{v}_{cm} \cdot \vec{v}_i^r + v_i^r)$$

Total energy summed over all particles:
$$K = \sum_i K_i = \sum_i \left( \frac{1}{2} m_i v_{cm}^2 \right) + \sum_i (m_i \vec{v}_{cm} \cdot \vec{v}_i^r) + \sum_i \left( \frac{1}{2} m_i v_i^r \right)$$

Taking common factors outside sum
$$K = \frac{1}{2} \left( \sum_i m_i \right) v_{cm}^2 + \vec{v}_{cm} \cdot \sum_i (m_i \vec{v}_i^r) + \sum_i \left( \frac{1}{2} m_i v_i^r \right)$$

Additivity of energies

Total energy
$$K = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2$$

Total kinetic energy of a rigid body can be divided into translational and rotational parts
Summary

Torque
\[ \tau = rF \sin(\phi) \]
\[ \vec{\tau} = \vec{r} \times \vec{F} \]

Newton’s second law for rotation
\[ \vec{\tau}_{\text{net}} = I \ddot{\alpha} \quad \text{or} \quad \tau_{\text{net}} = I \alpha \]

Work and power
\[ W = \tau(\theta_2 - \theta_1) \quad P = \tau \omega \]

Total kinetic energy
\[ K = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2 \]

Translational and rotational dynamics
\[ \vec{F}_{\text{net}} = m \ddot{x}_{cm} \quad \text{and} \quad \vec{\tau}_{\text{net}} = I_{cm} \ddot{\omega} \]

Angular momentum: particle
\[ \vec{L} = \vec{r} \times \vec{p} \quad \text{rigid body} \quad \vec{L} = I \vec{\omega} \]

Conservation of angular momentum
for \( \tau_{\text{net}} = 0 \), \( \vec{L} = \text{const} \)

correspondence for translational and rotational motion

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<td>Power (constant force)</td>
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more correspondence

<table>
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<tr>
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<td>Force</td>
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<td>$\vec{P} = \sum \vec{p}_i$</td>
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<td>Linear moment $^b$</td>
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<td>$\vec{P} = M\vec{v}_{com}$</td>
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<td>Newton’s second law $^b$</td>
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<td>$\vec{F}_{\text{net}} = \frac{d\vec{P}}{dt}$</td>
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<td>Conservation law $^d$</td>
<td>Conservation law $^d$</td>
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<tr>
<td>$\vec{p} = $ a constant</td>
<td>$\vec{L} = $ a constant</td>
</tr>
</tbody>
</table>

$^b$ for a system of particles

$^c$ for a rigid body about a fixed axis

$^d$ for a closed isolated system

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**Torque**

A square metal plate 0.18 m on each side is pivoted about an axis through point $O$. Calculate the net torque about this axis due to the three forces $F_1=$18.0N, $F_2=$26.0N, $F_3=$14N. The plate and all forces are in the plane of the page.
**Torque and angular acceleration**

A uniform, $M$ kg, spherical shell of radius $R$ has four small $m$ kg masses attached to its outer surface and equally spaced around. What frictional torque is needed to reduce its angular speed from $\omega_1$ to $\omega_2$ in $\Delta t$ seconds?

\[ \tau_{net} = I \alpha \]

Angular acceleration can be found from \( \alpha = \frac{\Delta \omega}{\Delta t} \)

Total inertia of the system \( I = I_{shell} + I_m \)

Where \( I_{shell} = \frac{2}{3} MR^2 \) and for the left and right \( I_m = mR^2 \)

\[ \tau_{net} = \left( \frac{2}{3} M + 2m \right) R^2 \frac{\Delta \omega}{\Delta t} \]

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**Which is fastest?**

- hoop
- empty can
- solid cylinder
- sphere
- box
Rolling or Sliding down a Ramp

Rolling a uniform body of mass $M$ and radius $R$

Forces and acceleration

- $\vec{F}_g = mg$ gravitational force
- $\vec{N}$ normal force
- $\vec{f}_s$ static frictional force
- $\vec{a}$ acceleration (center-of-mass)
- $\vec{\alpha}$ angular acceleration

Equations of motion

$$\begin{cases} 
Mg \sin \theta - f_s = Ma \\
Rf_s = I \alpha 
\end{cases}$$
Rolling or Sliding down a Ramp (cont.)

since the body is rolling smoothly \( a = \alpha R \) then
\[
Rf_s = I \frac{a}{R} \quad \text{or} \quad f_s = I \frac{a}{R^2}
\]
and the first equation is
\[
mg \sin \theta = I \frac{a}{R^2} + ma
\]
solving for \( a \) gives
\[
a = \frac{g \sin \theta}{1 + I/mR^2}
\]
for many rolling objects rotational inertia \( I = \beta mR^2 \) then
\[
a = \frac{g \sin \theta}{1 + \beta} \quad \text{and} \quad f_s = \frac{\beta}{1 + \beta} gm \sin \theta
\]
for a solid sphere \( I = \frac{2}{5} mR^2 \) (\( \beta = 2/5 \)) and \( a = \frac{5}{7} g \sin \theta \)

Rolling or Sliding down a Ramp (cont.)

the static friction force prevents the object from slipping
then
\[
f_s = \frac{\beta}{1 + \beta} gm \sin \theta = \mu_s mg \cos \theta
\]
the coefficient of static friction must be at least (or greater)
\[
\mu_s = \frac{\beta}{1 + \beta} \tan \theta
\]
to prevent slipping
for a solid sphere \( I = \frac{2}{5} mR^2 \) (\( \beta = 2/5 \)) and \( \mu_s = \frac{2}{7} \tan \theta \)
Rolling or Sliding down a Ramp (cont.)

Speed at the end of a ramp of length $d$

**Method 1:** dynamics of motion

for linear motion $v_f^2 - v_i^2 = 2ad$

for rolling dynamics $a = \frac{g \sin \theta}{1 + \beta}$

then $v_f^2 = 2d \frac{g \sin \theta}{1 + \beta}$

for a solid sphere $I = \frac{2}{5} mR^2$ ($\beta = 2/5$)

and $v_f^2 = \frac{10}{7} gd \sin \theta$

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Rolling or Sliding down a Ramp (cont.)

Speed at the end of a ramp of length $d$

**Method 2:** conservation of energy

$E_i = U_i = mgd \sin \theta$

$E_f = K_{\text{trans.}} + K_{\text{rot}}$

from conservation of energy

$mgd \sin \theta = \frac{mv_f^2}{2} + \frac{1}{2} I_0 \omega^2 = \frac{mv_f^2}{2} + \frac{1}{2} \beta mr^2 \frac{v_f^2}{R^2}$

$v_f^2 = \frac{2gd \sin \theta}{1 + \beta}$

for a solid sphere $I = \frac{2}{5} mR^2$ ($\beta = 2/5$) and $v_f^2 = \frac{10}{7} gd \sin \theta$
Rolling or Sliding down a Ramp (cont.)

Race of a rolling body and a sliding box (same mass)

Using conservation of energy
For a box sliding down a ramp with friction

\[ mgd \sin \theta = \frac{mv_f^2}{2} - \mu_k mg \cos \theta \quad \text{and then} \quad v_f^2 = 2gd(\sin \theta - \mu_k \cos \theta) \]

Normally for coefficients of friction \( \mu_k \leq \mu_s \)

For a rolling sphere we have \( \mu_s = \frac{2}{7} \tan \theta \)

If \( \mu_k = \mu_s \) then \( v_f^2 = \frac{10}{7} gd \sin \theta \) (same speed as a sphere)

If \( \mu_k < \mu_s \) then \( v_f^2 > \frac{10}{7} gd \sin \theta \) (faster comparing to a sphere)

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Torque and angular acceleration

A box of mass \( m_1 \) resting on a horizontal frictionless surface is attached to a box of mass \( m_2 \) by a thin cord. The pulley has the shape of a uniform solid disk of mass \( M \) and radius \( R \). After the system is released find

a) the tension in the cord on both sides of the pulley

b) the acceleration of the box
Torque and angular acceleration

Free-body diagrams

\[ T_1 = m_1 a \]
\[ (T_2 - T_1)R = \frac{1}{2} MR^2 \frac{a}{R} \]
\[ m_2 g - T_2 = m_2 a \]

three unknowns \( a, T_1, T_2 \) and three linear equations

solution for acceleration \( a = \frac{m_2 g}{m_1 + m_2 + M / 2} \)