Geometric optics and rainbows: generalization of a result by Huygens

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In 1652 Huygens derived a formula specifying the rainbow angle for the primary bow ($k = 1$) in terms of the refractive index only. A generalization of this result for any $k \geq 1$ is outlined, along with an alternative representation. The details of the derivation can be found in (Adam, Mathematics Magazine, 2008, under review), but the results as stated may be of interest to the atmospheric optics community. © 2008 Optical Society of America

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It is well known that for k internal reflections and two refractions in a spherical raindrop, the angle through which an incoming ray is deviated (as a function of the angle of incidence i) is defined by

$$D_k(i) = k\pi + 2i - 2(k+1)\arcsin\left(\frac{\sin i}{n}\right), \qquad (1)$$

$n > 1$ being the constant refractive index of the drop. The reason a rainbow exists at all (and would do even in the absence of spectral dispersion, as a "whitebow") is because there is an extremum (a minimum to be precise) in $D_k(i)$ at the specific angle of incidence i_c defined by $dD_k(i)/di = 0$. This defines the direction of the "rainbow angle," i.e., the angle, relative to the original ray direction, through which the ray is deviated. Again, it is a standard exercise to establish that

$$i_c = \arccos\left[\frac{n^2 - 1}{k(k+2)}\right]^{1/2}. \qquad (2)$$

In 1652 Christian Huygens derived a result for $D_1(i_c)$ for $k = 1$ as a function of the refractive index n only, i.e., without any reference to i or i_c (stated in [1], though no reference is provided). Thus he found that

$$D_1(i_c) = 2\arccos\left[\frac{1}{n^2}\left(\frac{4 - n^2}{3}\right)^{3/2}\right]. \qquad (3)$$

For $n = 4/3$, $D_1(i_c) \approx 138°$. It is of interest to express the deviation $D_k(i_c)$ in similar terms. This has been carried out elsewhere [2], and although it is algebraically somewhat intensive, it can be expressed in this form for any positive integer k. In all the results that follow, $D_k(i_c)$ is expressed for simplicity as a positive angle less than 180°. This is not a concern when one recognizes that the incoming ray defines an axis of symmetry for the spherical raindrop, and so the results are cylindrically symmetric, and the sign of $D_k(i_c)$ is immaterial.

To illustrate the procedure for $k = 1$ it follows from Eqs. (1) and (2) that

$$\frac{1}{2}(D_1(i_c) - \pi) = \arccos\left(\frac{n^2 - 1}{3}\right)^{1/2} - 2\arcsin\left(\frac{4 - n^2}{3n^2}\right)^{1/2} \equiv A - 2B. \qquad (4)$$

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This is readily generalized to arbitrary values of $k \geq 1$, and elementary trigonometry yields the following results:

$$\sin A = \left[\frac{(k+1)^2 - n^2}{k(k+2)}\right]^{1/2},$$
$$\cos A = \left[\frac{n^2 - 1}{k(k+2)}\right]^{1/2},$$
$$\sin B = \left[\frac{(k+1)^2 - n^2}{n^2 k(k+2)}\right]^{1/2},$$
$$\cos B = \left[\frac{(k+1)^2 (n^2 - 1)}{n^2 k(k+2)}\right]^{1/2}. \tag{5}$$

Proceeding as with the case $k = 1$, it can be shown that

$$D_2(i_c) = 2 \arcsin\left[(n^2 - 1)^{1/2} \left(\frac{(9 - n^2)^{1/2}}{2n}\right)^3\right], \tag{6}$$

$$D_3(i_c) = 2 \arccos\left[\frac{1}{5(15)^{3/2}} \frac{(16 - n^2)^{3/2}}{n^4} (27n^2 - 32)\right]. \tag{7}$$

(Refer to the Appendix for a summary of the procedure.) One can verify that $D_2(i_c) \approx 129°$ and $D_3(i_c) \approx 42°$ for $n = 4/3$, corresponding to the value obtained using Eqs. (1) and (2). The objective of this analysis was to express the rainbow angle for higher-order rainbows explicitly in the form of Huygens' original result, i.e., as a single angle expressed in terms of the refractive index. The generalization of specific results is a common mathematical endeavor, not for aesthetic reasons only, of course, but to express results in readily applicable yet compact form. However, as will be observed below, the cases for $k = 4$ and $k = 5$ become increasingly inelegant, compact mathematical form notwithstanding. Specifically, expressions for these cases are as follows:

$$D_4(i_c) = 2 \arcsin\left[\frac{(n^2 - 1)^{1/2} (25 - n^2)^{3/2} (25 - 16n^2)}{6^3 n^5}\right], \tag{8}$$

$$D_5(i_c) = 2 \arccos\left[\frac{(36 - n^2)^{3/2} (9792n^2 - 3125n^4 - 6912)}{7^{7/2} 5^{5/2} n^6}\right]. \tag{9}$$

Again, for $n = 4/3$, these formulas yield the results $D_4(i_c) \approx 44°$ and $D_5(i_c) \approx 128°$. The results for arbitrary positive values of k are stated in the Appendix and proved elsewhere [2].

An alternative result, expressing the rainbow angle (in terms of n alone) as the *difference* of two angles follows from substituting Eq. (2) into Eq. (1) and simplifying the result, i.e.,

$$D_k(i_c) = k\pi + 2 \arccos\left(\frac{n^2 - 1}{k(k+2)}\right)^{1/2} - 2(k+1) \arcsin\left(\frac{1}{n^2} - \left(\frac{n^2 - 1}{k(k+2)n^2}\right)\right)^{1/2}. \tag{10}$$

This equation is actually used to generate the forms in Eqs. (3) and (6)–(9) above. Thus, in contrast to those equations, we obtain the set

$$D_1(i_c) = \pi + 2 \arccos\left(\frac{n^2 - 1}{3}\right)^{1/2} - 4 \arcsin\left(\frac{1}{n^2} - \left(\frac{n^2 - 1}{3n^2}\right)\right)^{1/2},$$

$$D_2(i_c) = 2\pi + 2 \arccos\left(\frac{n^2 - 1}{8}\right)^{1/2} - 6 \arcsin\left(\frac{1}{n^2} - \left(\frac{n^2 - 1}{8n^2}\right)\right)^{1/2},$$

$$D_3(i_c) = 3\pi + 2 \arccos\left(\frac{n^2 - 1}{15}\right)^{1/2} - 8 \arcsin\left(\frac{1}{n^2} - \left(\frac{n^2 - 1}{15n^2}\right)\right)^{1/2},$$

$$D_4(i_c) = 4\pi + 2 \arccos\left(\frac{n^2 - 1}{24}\right)^{1/2} - 10 \arcsin\left(\frac{1}{n^2} - \left(\frac{n^2 - 1}{24n^2}\right)\right)^{1/2},$$

$$D_5(i_c) = 5\pi + 2 \arccos\left(\frac{n^2 - 1}{35}\right)^{1/2} - 12 \arcsin\left(\frac{1}{n^2} - \left(\frac{n^2 - 1}{35n^2}\right)\right)^{1/2}.$$

For $n = 4/3$ and $k = 1, 2, 3, 4$ and 5, respectively, Eq. (10) yields the same results as before (again expressed as an angle less than 180°), as is readily verified. Formulas equivalent to Eq. (10), but expressed in terms of inverse sine functions, can be found in [3], and (implicitly) in terms of inverse tangent functions, in [4]. Values of $D_k(i_c)$ for k values up to six can also be found in [5] (though not expressed as angles less than 180°).

Appendix: $D_k(i_c)$ for Arbitrary Values of the Integer k

Evaluating $\sin[(D_k(i_c) - k\pi)/2]$ from Eq. (10), it can be shown that [2]

$$(-1)^{k/2} \sin \frac{D_k(i_c)}{2} = \left[\frac{(k+1)^2 - n^2}{k(k+2)}\right] \times \cos[(k+1)\left(\arccos\left\{\frac{(k+1)^2(n^2-1)}{n^2k(k+2)}\right\}^{1/2}\right)] - \left[\frac{n^2-1}{k(k+2)}\right] \sin\left[(k+1) \times \left(\arccos\left\{\frac{(k+1)^2(n^2-1)}{n^2k(k+2)}\right\}^{1/2}\right)\right] \quad \text{(A1)}$$

for even values of k. For odd values of k, the left-hand side of Eq. (A1) is replaced by the expression $(-1)^{(k+1)/2} \cos \frac{D_k(i_c)}{2}$.

To obtain a more succinct representation, we can denote the right-hand side of Eq. (A1) by $\Phi(k, n)$. It follows that for even k values, Eq. (A1) can be rewritten as

$$|D_k(i_c)| = 2(-1)^{k/2} \arcsin[\Phi(k, n)] \quad \text{(A2)}$$

and for odd k values

$$|D_k(i_c)| = 2 \arccos[\Phi(k, n)]. \quad \text{(A3)}$$

Note that the ranges of the inverse sine function and inverse cosine function, in degrees, are $[-90°, 90°]$ and $[0°, 180°]$ respectively.

It would be interesting to try and find a general pattern for the $D_k(i_c)$ based on Eqs. (3) and (6)–(9), or equivalently, on the arguments of the trigonometric functions in Eq. (A1). Certainly, in the former the factor $[((k+1)^2 - n^2)n^{-(k+1)}]^{3/2}$ appears, along with (for odd values of k) the denominator $(k+2)^{(k+2)}k^k$. In Eq. (A1), the existence of the angles $(k+1)\alpha$, where

$$\alpha = \arccos\left\{\frac{(k+1)^2(n^2-1)}{n^2k(k+2)}\right\}^{1/2} \quad \text{(A4)}$$

indicates that the expansions of $\sin[(k+1)\alpha]$ and $\cos[(k+1)\alpha]$ might be valuable in this regard, using either the multiple angle formulas, or in terms of Chebyshev polynomials of the first and second kind [6]. However, such further investigations are outside the scope of this paper.

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