Electromagnetism HW 1 – math review

Problems 1-5 due Mon 7th Sep, 6-11 due Mon 14th Sep

Exercise 1. The Levi-Civita symbol, ϵ_{ijk}, also known as the completely antisymmetric rank-3 tensor, has the following properties:
(a) $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1$
(b) $\epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1$
(c) all other entries are zero.
It follows that $\epsilon_{jik} = -\epsilon_{ijk}$ and similar antisymmetric relations hold.

1.1 Using a notation in which repeated indices are summed over, show that

(a) $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$,

(b) $\epsilon_{ijk}\epsilon_{ijm} = 2\delta_{km}$, and

(c) $\epsilon_{ijk}\epsilon_{ijk} = 6$,

where the Kronecker delta, δ_{ij}, is equal to $+1$ if $i = j$ and zero if $i \neq j$.

1.2 Show that $C_i = \epsilon_{ijk}A_jB_k$ has the same components as the vector $\vec{C} = \vec{A} \times \vec{B}$

1.3 Using properties of the Levi-Civita symbol, prove that

(a) $\vec{A} \times \vec{B} \times \vec{C} = (\vec{A} \cdot \vec{C})\,\vec{B} - (\vec{A} \cdot \vec{B})\,\vec{C}$,

(b) $(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$,

(c) $(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C} \times \vec{D})\vec{B} - (\vec{B} \cdot \vec{C} \times \vec{D})\vec{A}$,

(d) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{g}) = 0$,

(e) $\vec{\nabla} \times (\vec{\nabla} \times \vec{g}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{g}) - \nabla^2\vec{g}$,

(f) $\vec{\nabla} \cdot (f\vec{g}) = f\,\vec{\nabla} \cdot \vec{g} + \vec{g} \cdot \vec{\nabla}f$,

(g) $\vec{\nabla} \times (f\vec{g}) = f\,\vec{\nabla} \times \vec{g} - \vec{g} \times \vec{\nabla}f$,

(h) $\vec{\nabla} \cdot (\vec{f} \times \vec{g}) = \vec{g} \cdot (\vec{\nabla} \times \vec{f}) - \vec{f} \cdot (\vec{\nabla} \times \vec{g})$,

(i) $\vec{\nabla} \times (\vec{f} \times \vec{g}) = \vec{f}(\vec{\nabla} \cdot \vec{g}) - \vec{g}(\vec{\nabla} \cdot \vec{f}) - (\vec{f} \cdot \vec{\nabla})\vec{g} + (\vec{g} \cdot \vec{\nabla})\vec{f}$, and

(j) $\vec{\nabla} \times (\vec{g} \times \vec{r}) = 2\vec{g} + r\frac{\partial}{\partial r}\vec{g} - \vec{r}(\vec{\nabla} \cdot \vec{g})$, where $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

Exercise 2. The determinant of the Jacobian matrix, $\mathbf{J}$, relates volume elements when changing variables in an integral. For example if we change from variables $\vec{x} = (x_1, x_2 \ldots x_N)$ to variables $\vec{y} = (y_1, y_2 \ldots y_N)$, the volume elements are related by

$$d^N x = \left|\mathbf{J}(\vec{x}, \vec{y})\right| d^N y = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_N} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_N} \\ & \vdots & & \\ \frac{\partial x_N}{\partial y_1} & \frac{\partial x_N}{\partial y_2} & \cdots & \frac{\partial x_N}{\partial y_N} \end{vmatrix} d^N y.$$

Starting from the volume element in cartesian coordinates, $dx\, dy\, dz$, use the Jacobian to show that the volume element is

(a) $\rho\, d\rho\, d\phi\, dz$ in cylindrical coordinates, and

(b) $r^2 \sin\theta\, dr\, d\theta\, d\phi$ in spherical coordinates.

Exercise 3. Starting from the divergence theorem,

$$\int_V d^3 r\, \vec{\nabla} \cdot \vec{F} = \int_S d\vec{S} \cdot \vec{F},$$

show that

(a) $\int_V d^3 r\, \vec{\nabla} \phi = \int_S d\vec{S}\, \phi$ for a scalar field $\phi(\vec{r})$
[*Hint: choose* $\vec{F} = \vec{c}\, \phi(\vec{r})$ *with a constant vector* $\vec{c}$],

(b) $\int_V d^3 r\, \vec{\nabla} \times \vec{A} = \int_S d\vec{S} \times \vec{A}$ for a vector field $\vec{A}(\vec{r})$
[*Hint: choose* $\vec{F} = \vec{A}(\vec{r}) \times \vec{c}$ *with a constant vector* $\vec{c}$],

(c) $\int_V d^3 r \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi\right) = \int_S d\vec{S} \cdot \left(\phi \vec{\nabla} \psi - \psi \vec{\nabla} \phi\right)$,

(d) for a closed surface, S, enclosing a volume, V, $\int_S d\vec{S} = 0$, and $\int_S d\vec{S} \cdot \vec{r} = 3V$.

Exercise 4. Show that a vector field with zero curl can be expressed as the gradient of a scalar field, $\vec{F}(\vec{r}) = \vec{\nabla}\Phi(\vec{r})$.

Show that the line integral, $\int_{\vec{r}_i}^{\vec{r}_f} d\vec{\ell} \cdot \vec{F}$, of a curl-less vector field, $\vec{F}$, between two points $\vec{r} = \vec{r}_i$ and $\vec{r} = \vec{r}_f$ is independent of the path taken between the two points.

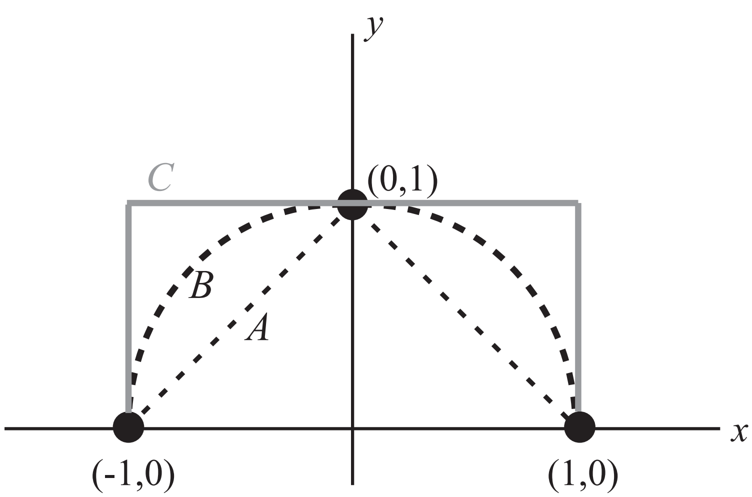
For the particular field

$$\vec{F} = \frac{-y\hat{x} + x\hat{y}}{x^2 + y^2},$$

show that

(a) $\vec{\nabla} \times \vec{F} = 0$,

(b) the line integral between $(x = -1, y = 0)$ and $(x = 1, y = 0)$ has value $-\pi$ for each of the paths, A, B, C, shown in the figure.



(c) Find a scalar field, $\Phi(x, y)$, such that $\vec{F} = \vec{\nabla}\Phi$.

[*Hint: integrate* $\vec{F} \cdot d\vec{\ell}$ *from* $(0, 0)$ *to* (x, y) *choosing a path that makes the integrals as simple as possible*]

Exercise 5. Evaluate $\int_S d\vec{S} \cdot \vec{F}$ where

$$\vec{F} = (y^2 - x^2)\hat{x} + (2xy - y)\hat{y} + 3z\hat{z}$$

for the entire surface of the tin can bounded by the cylinder $x^2 + y^2 = 16$, $z = -3$, $z = 3$,

(a) by explicitly computing the surface integral, and

(b) by using the divergence theorem.

Exercise 6. Consider the following vector field, expressed in cylindrical coordinates, $\vec{W}(\rho, \phi, z) = \frac{\alpha}{\rho}\hat{\phi}$. Show that the z-component of the curl is zero everywhere except at the origin and

$$\left(\vec{\nabla} \times \vec{W}\right)_z = \frac{\alpha}{\rho}\delta(\rho)$$

Exercise 7. In lectures we showed, using the divergence theorem, that

$$\nabla^2 \frac{1}{r} = -4\pi\delta(\vec{r}).$$

Let's explore another way of deriving the same result:

(a) Show that the function $\Lambda(r, a) = \nabla^2 \frac{1}{\sqrt{r^2+a^2}}$ is $\Lambda(r, a) = \frac{-3a^2}{(r^2+a^2)^{5/2}}$, and plot $\Lambda(r, a)$ versus r for a range of values of a decreasing toward zero.

(b) Show that the integral over all space of $\Lambda(r, a)$ is of value -4π.

(c) Explain how we can conclude that $\lim_{a\to 0} \Lambda(r, a) = -4\pi\delta(\vec{r})$

Exercise 8. The Helmholtz theorem for a divergence-less and curl-less vector field: Suppose a vector field satisfies $\vec{\nabla} \cdot \vec{E} = 0$ and $\vec{\nabla} \times \vec{E} = 0$ everywhere in a volume, V, bounded by a surface, S. Use a derivation like the one we used in lecture to show that $\vec{E}(\vec{r})$ can be found everywhere in V if its value is known at all points on the surface, S.

Notice that your result shows that if $\vec{E}$ is zero everywhere on the boundary, then $\vec{E}$ is also zero everywhere inside the volume – this will be realized when we consider an empty cavity in a perfect conductor.

Exercise 9. The **Legendre differential equation** is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \ell(\ell+1)y = 0.$$

One set of solutions to this equation, when ℓ takes positive integer values, are known as the Legendre polynomials, $y(x) = P_\ell(x)$. They are polynomials of order ℓ that are normalized such that $P_\ell(1) = 1$. The first few are $P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2-1),\ P_3(x) = \frac{1}{2}(5x^3-3x), \ldots$

9.1 Sketch graphs of $P_0(x), P_1(x), P_2(x), P_3(x)$ in the range $-1 \leq x \leq 1$.

9.2 We can prove that the solutions to Legendre's equation also satisfy Rodrigues' formula,

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell.$$

(a) Check that you can obtain $P_0(x) \ldots P_3(x)$ using Rodrigues' formula.

(b) Using Rodrigues' formula show that $\int_{-1}^{+1} dx\, x^m P_\ell(x) = 0$ if $m < \ell$.

9.3 The "generating function" for Legendre polynomials is

$$\Phi(x,h) = \frac{1}{\sqrt{1-2xh+h^2}} = \sum_{\ell=0}^{\infty} h^\ell P_\ell(x).$$

(a) Show that the functions $P_\ell(x)$ in the sum here do indeed satisfy Legendre's equation and have the property $P_\ell(1) = 1$. [*Hint:* consider $\frac{\partial\Phi}{\partial x}$, $\frac{\partial^2\Phi}{\partial x^2}$ and $h\frac{\partial^2}{\partial h^2}(h\Phi)$]

(b) Prove the identity $xP'_\ell(x) - P'_{\ell-1}(x) = \ell P_\ell(x)$. [*Hint:* consider $\frac{\partial\Phi}{\partial x}$ and $\frac{\partial\Phi}{\partial h}$]

9.4 The Legendre polynomials form a complete orthogonal basis with the property that

$$\int_{-1}^{+1} dx P_\ell(x) P_m(x) = \delta_{m\ell}\, N_\ell$$

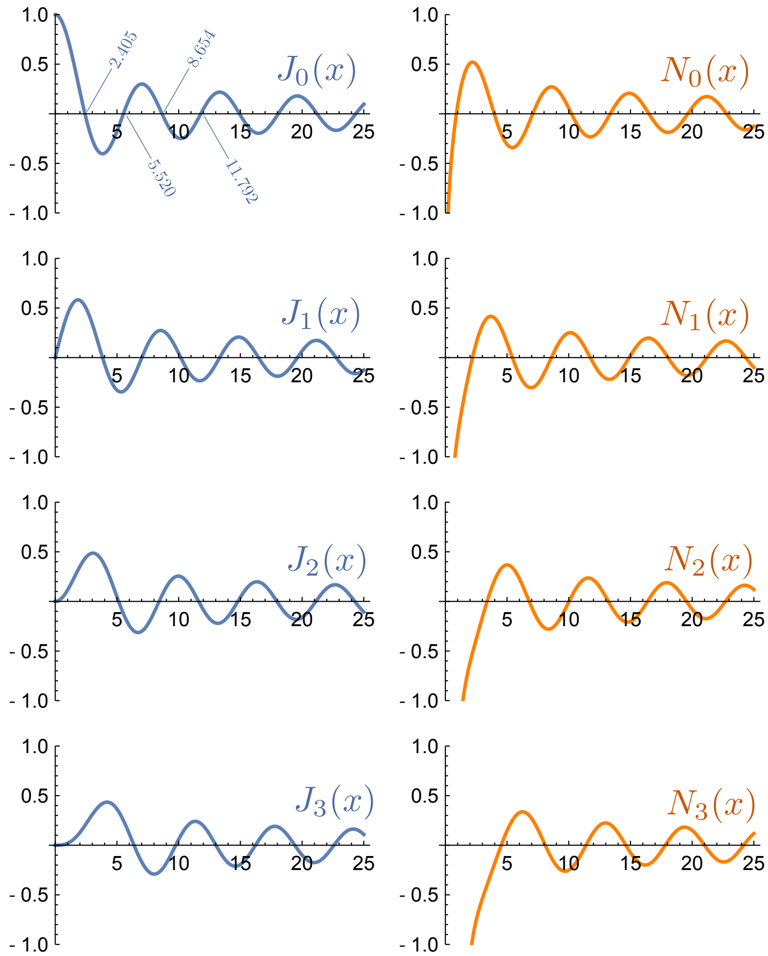
(a) Show that $N_\ell = \frac{2}{2\ell+1}$ using the identity derived in **9.3(b)** and the result obtained in **9.2(b)**.

(b) Prove that the $P_\ell(x)$ are complete on the interval $-1 \leq x \leq 1$ by showing that $D(x,x') = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} P_\ell(x)P_\ell(x')$ is a representation of the Dirac delta function, $D(x,x') = \delta(x-x')$. [*Hint:* Start from an arbitrary function $f(x)$ expanded as a infinite superposition of $P_\ell(x)$ and then show that $\int_{-1}^{+1} dx\, D(x,x') f(x) = f(x')$]

Exercise 10. The **Bessel differential equation** is

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - p^2) y = 0.$$

If we restrict ourselves to $x \geq 0$ and integer values of p, the linearly independent solutions to this equation are known as Bessel functions, $J_p(x)$, and Neumann functions, $N_p(x)$. The first few such functions are plotted below. Note that the Neumann functions diverge at the origin.



10.1 Bessel functions are orthogonal, but in a way that might look strange:

$$\int_0^1 dx\, x\, J_p(a_n x)\, J_p(a_m x) = \delta_{n,m}\, N_n$$

In this equation, a_n and a_m are positions of zeros of the Bessel function, i.e. $J_p(a_n) = 0$. Prove the orthogonality expression by taking the following steps:

(a) Show that $J_p(ax)$ satisfies the equation

$$x \frac{d}{dx}\left(x \frac{d}{dx} J_p(ax) \right) + (a^2 x^2 - p^2) J_p(ax) = 0.$$

(b) Show that

$$J_p(bx)\,\frac{d}{dx}\left(x\frac{d}{dx}J_p(ax)\right) - J_p(ax)\,\frac{d}{dx}\left(x\frac{d}{dx}J_p(bx)\right) + (a^2-b^2)\,x\,J_p(ax)\,J_p(bx) = 0,$$

and integrate this expression in the case that a and b are chosen to be two different zeroes of the Bessel function $J_p(x)$ to obtain the orthogonality relation.

(c) Finally, show that $N_n = \frac{1}{2}\big(J_p'(a_n)\big)^2$ [*Hint:* you might want to consider the proof in (b) but only assuming that b is a zero of J_p and not a, and be careful taking the limit $a \to b$].

10.2 Bessel functions get used when we solve problems with cylindrical symmetry. Consider the following functions of cylindrical coordinates, ρ, z (and independent of ϕ),

$$f_1(\rho, z) = \frac{1}{\sqrt{\rho^2+z^2}},$$
$$f_2(\rho, z) = \int_0^\infty dk\, A(k) J_0(k\rho) e^{-k|z|}.$$

(a) Show that these functions both satisfy Laplace's equation $\nabla^2 f = 0$.

(b) Assuming that $f_1 = f_2$, by considering $\rho = 0$ show that $A(k) = 1$ and thus that,

$$\frac{1}{\sqrt{\rho^2+z^2}} = \int_0^\infty dk\, J_0(k\rho) e^{-k|z|},$$

which is known as a Fourier-Bessel representation of $\frac{1}{\sqrt{\rho^2+z^2}}$.

Exercise 11. Functions can often usefully be expanded as superpositions of orthogonal basis functions, for example as **Fourier series** and **transforms**.

11.1 Show that the function $f(x) = |x|$ on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$ has a Fourier series representation

$$\frac{\pi}{4} - \frac{2}{\pi}\sum_{\text{odd}\, n} \frac{1}{n^2}\cos 2nx$$

11.2 Show that the function $f(x) = \begin{cases} 1, & -1 < x < 1 \\ 0, & |x| > 1 \end{cases}$ has a Fourier representation

$$f(x) = \frac{2}{\pi}\int_0^\infty dk\, \frac{\sin k}{k}\cos kx.$$

Using the result that $\int_{-\infty}^{\infty} dy\, \frac{\sin y}{y} = \pi$, explicitly compute the above integral in the cases $|x| < 1$ and $|x| > 1$ and check that you get 1 and 0 respectively.