Electromagnetism HW 2
Charge distributions, Gauss's law & Energy

All problems due Mon 21st Sep

Exercise 1. Express the following charge distributions as three-dimensional charge densities $\rho(\vec{r})$, using Dirac delta functions where necessary.

1.1 In spherical coordinates, a charge Q uniformly distributed over a spherical shell of radius R.

1.2 In cylindrical coordinates, a charge λ per unit length uniformly distributed over a cylindrical surface of radius b.

1.3 In cylindrical coordinates, a charge Q spread uniformly over a flat circular sheet of radius R.

1.4 The same as part **1.3**, but using spherical coordinates.

Exercise 2. In this problem we will make use of the superposition of charges,

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3\vec{r}\,'\rho(\vec{r}\,')\,\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3},$$

to find the electric field due to continuous charge distributions.

2.1 Show that the electric field on the symmetry axis of a ring of radius R with a uniform charge per unit length of λ is,

$$\vec{E}(z) = \frac{Q}{4\pi\epsilon_0}\frac{z}{(z^2+R^2)^{3/2}}\hat{z},$$

where we call the distance from the plane of the ring, z, and the total charge is $Q = \lambda \cdot 2\pi R$.

2.2 Show, by superposition of charges, that the electric field on the symmetry axis of a disk of radius R with uniform charge per unit area σ is,

$$\vec{E}(z) = \frac{\sigma}{2\epsilon_0}\left(\mathrm{sgn}(z) - \frac{z}{(z^2+R^2)^{1/2}}\right)\hat{z},$$

where $\text{sgn}(z) = z/|z|$.

2.3 Find the electric field on the symmetry axis of a disk of radius R with uniform charge per unit area σ but having a circular hole in the middle of radius a.

2.4 Use the result of part **2.2** to find the electric field from an infinite sheet with uniform charge per unit area of σ.

2.5 Use the result of part **2.3** to find the electric field from an infinite sheet with uniform charge per unit area of σ having a circular hole of radius a. You need only find the field on the symmetry axis.

2.6 Show that in cases **2.2** and **2.4** the field is discontinuous by an amount σ/ϵ_0 at $z = 0$, while in cases **2.4**, **2.5**, the field is continuous.

2.7 Find $\vec{E}(\vec{r})$ inside and outside a uniformly charged spherical shell by superposing the electric fields produced by a collection of charged rings (part **2.1**). [*Hint:* An ring of infinitesimal thickness at angular position, θ, will contain infinitesimal charge, $dQ = \sigma(2\pi R \sin\theta)(R d\theta)$]

Obtain the same results using Gauss's law.

2.8 Find $\vec{E}(\vec{r})$ inside and outside a uniformly charged spherical volume by superposing the electric fields produced by a collection of charged disks (part **2.2**). [*Hint:* A disk of infinitesimal thickness at angular position, θ, has an infinitesimal surface charge density of $d\sigma = \rho\, R \sin\theta d\theta$]

Obtain the same results using Gauss's law.

Exercise 3. A static charge distribution produces a spherically radial electric field,

$$\vec{E}(\vec{r}) = A\, e^{-\beta r}\, \frac{\vec{r}}{r^2},$$

with A and β being positive constants. Find the charge density, $\rho(r)$, which gives this electric field. Sketch $\rho(r)$, and find the total charge.

Exercise 4. Consider four identical positive point charges, Q, located at the following positions

$$\vec{r}_1 = [+1, +1, 0]; \quad \vec{r}_2 = [-1, +1, 0]; \quad \vec{r}_3 = [-1, -1, 0]; \quad \vec{r}_1 = [+1, -1, 0].$$

4.1 If a positive test particle of charge q is placed at the origin, show that is feels zero force from the four charges.

4.2 Show that near the origin, the potential from the four charges takes the form $\varphi_0 + Ax^2 + By^2 + Cz^2$ and determine the constants, φ_0, A, B, C.

4.3 Discuss whether the charge q is in stable equilibrium.

Exercise 5. Consider a uniform sphere of charge having total charge Q and radius R.

5.1 Using Gauss's law show that the electric field is

$$\vec{E}(\vec{r}) = \begin{cases} \hat{r}\frac{Q}{4\pi\epsilon_0}\frac{1}{r^2} & \text{for} \quad r > R \\ \hat{r}\frac{Q}{4\pi\epsilon_0}\frac{r}{R^3} & \text{for} \quad r < R \end{cases}.$$

5.2 Find the potential everywhere (note that $\phi(r \to \infty) \to 0$).

5.3 Show that the total energy needed to assemble this charge is

$$\frac{1}{4\pi\epsilon_0}\frac{3Q^2}{5R}$$

Exercise 6. Consider a spherical shell of radius R carrying a uniform charge. Show that the total energy needed to assemble this charge is

$$\frac{1}{4\pi\epsilon_0}\frac{Q^2}{2R}$$

if the total charge on the shell is Q.

Exercise 7. Let the space between two concentric spheres of radii a and $R > a$ be filled uniformly with charge.

7.1 Calculate the total energy. Check that the limits $a \to 0$ and $R \to a$ agree with the results of the previous two exercises.

7.2 Express your answer above in terms of the variable $x = a/R$. Show that $x \to 1$ minimizes the total energy for fixed total charge Q.

Exercise 8. We showed in class that the force per unit area on a surface charge density $\sigma(\vec{r}_S)$ is given by $\frac{d\vec{F}}{dS} = \frac{1}{2}\sigma(\vec{E}_1 + \vec{E}_2)$ where $\vec{E}_1$ and $\vec{E}_2$ are the electric fields on either side of the surface.

8.1 Find the total force exerted on itself by an infinite plane carrying uniform surface charge density, σ.

8.2 Find the total force exerted on itself by a spherical shell of radius R carrying uniform surface charge density, σ.