# Electromagnetism HW 3 – Multipoles

due Mon 28th Sept.

**Exercise 1.** Three point charges lie on the $z$-axis, a charge $+q$ at $z = +a$, a charge $-2q$ at $z = 0$ and a charge $+q$ at $z = -a$.

**1.1** Compute the multipole moments, $A_{\ell m}$, with $\ell \leq 2$, for this charge distribution.
[ *if you are able to, find an expression for* $A_{\ell m}$ *valid for all* $\ell$ ]

**1.2** Keeping only the lowest order term in the multipole expansion write an expression for the potential.

**1.3** By superposing the potentials from each charge, find the exact potential at any point in the $x - y$ plane. Show that the result obtained in **1.2** correctly describes the potential at distances from the charge distribution must larger than $a$.

**Exercise 2.** Consider a spherical shell of radius $R$ which carries a surface charge density of $\sigma(\theta, \phi) = \sigma_0 \sin\theta \cos\phi$.

**2.1** Find the exterior spherical multipole moments for this charge distribution.

**2.2** Find the potential outside the sphere.

**2.3** Find the potential inside the sphere.

**2.4** Check that the potential and electric field obey the correct matching conditions at the sphere.

**2.5** What is the electric dipole moment of the sphere ?

**Exercise 3.** Two point dipoles, moments $\vec{p}_1$ and $\vec{p}_2$, are separated by a distance $R$. Suppose $\vec{p}_1$ lies at the origin, and choose the $z$-axis in the direction of $\vec{p}_1$ and the $x$-axis to be along the line from $\vec{p}_1$ to $\vec{p}_2$. Suppose $\vec{p}_2$ lies in the $xz$-plane and makes an angle $\alpha$ with the $z$-axis.

Describe the force and torque on $\vec{p}_2$ due to the presence of $\vec{p}_1$. Is there a stable equilibrium orientation for $\vec{p}_2$ ?

You may use the results we obtained in class for
(a) the potential energy of a dipole $\vec{p}$ in an external field $\vec{E}$, $U = -\vec{p}\cdot\vec{E}$,
(b) the force on a dipole $\vec{p}$ in an external field $\vec{E}$, $\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E}$,
(c) the torque on a dipole $\vec{p}$ in an external field $\vec{E}$, $\vec{\tau} = \vec{p}\times\vec{E}$.

**Exercise 4.** An origin-centered uniform ring of charge lies in the $x-y$ plane with total charge $Q$ and radius $a$. A coplanar and concentric uniform ring of charge has radius $b > a$ and total charge $-Q$.

Using a multipole expansion, keeping only the lowest nonzero moment, find an expression for the potential valid at large distances from the rings.

**Exercise 5.** Interior and exterior spherical multipole expansions can be written

$$\begin{aligned}
\phi_{\text{ext.}}(\vec{r}) &= \frac{1}{4\pi\epsilon_0}\sum_{\ell,m} A_{\ell,m}\,\frac{1}{r^{\ell+1}}\,Y_{\ell m}(\theta,\phi) \\
\phi_{\text{int.}}(\vec{r}) &= \frac{1}{4\pi\epsilon_0}\sum_{\ell,m} B_{\ell,m}\,r^{\ell}\,Y^*_{\ell m}(\theta,\phi).
\end{aligned} \tag{1}$$

The Laplacian operator in spherical coordinates takes the form

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right].$$

Using the fact that the spherical harmonic functions satisfy the differential equation

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y_{\ell m}(\theta,\phi) + \ell(\ell+1)Y_{\ell m}(\theta,\phi) = 0,$$

show that each term in the expansions in Eq. 1 satisfies $\nabla^2\phi(\vec{r}) = 0$, and hence that the sums also satisfy Laplace's equation.

**Exercise 6.** In this problem we'll consider a continuous charge distribution which viewed on large scales corresponds to a dipole layer, and show that it gives rise to a change in the electric potential that viewed on large scales would appear to be a discontinuity.

**6.1** Plot the function

$$\rho(z) = \frac{1}{L^2} \frac{\sinh z/L}{\cosh^3 z/L}$$

for a range of $L$ values.

**6.2** Find the electric potential, $\phi(z)$, corresponding to the charge distribution given in **5.1** using Poisson's equation $\frac{d^2\phi}{dz^2} = -\rho/\epsilon_0$. Plot the potential for a range of $L$ values.

**6.3** Show that the derivative of $\frac{(2L)^{-1}}{\cosh^2 z/L}$ is $-\rho(z)$, and justify the claim that

$$\lim_{L \to 0} \frac{(2L)^{-1}}{\cosh^2 z/L} = \delta(z).$$

Show that the singularity in $\rho(z)$ as $L \to 0$ matches our description in lectures of a dipole layer.