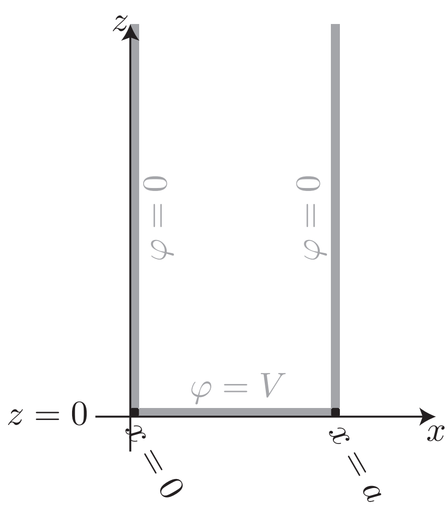
Electromagnetism HW 6 – finding potentials

1-4 due Mon 26th Oct, 5-8 due Mon 2nd Nov

Exercise 1. The figure shows a cross-section of an infinitely long and deep slot formed by two grounded conducting plates at $x = 0$ and $x = a$, and a conducting plate at $z = 0$ which is held at a potential V. Find the potential inside the slot (you may leave your answer in terms of a sum) and determine the asymptotic behavior for $z \gg a$.

[*Hint*: You can use separable solutions of Laplace's equation in Cartesian coordinates – choose the right ones for the boundary conditions you need to apply.]



Exercise 2. Find the volume charge density, ρ, and surface charge density, σ present in and on a sphere of radius R if the field produced by these charges inside the sphere is

$$\vec{E} = -2V_0 \frac{xy}{R^3}\hat{x} + V_0 \frac{y^2 - x^2}{R^3}\hat{y} - V_0 \frac{1}{R}\hat{z}.$$

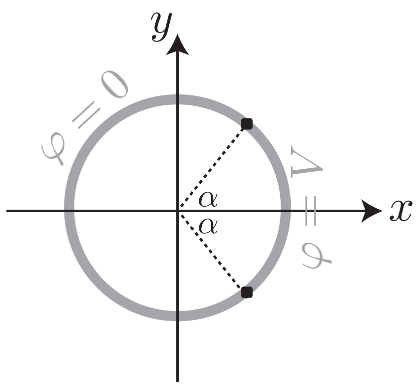
There is no charge anywhere outside the sphere. Express your answer as a function of the angles θ, ϕ describing position on the sphere with respect to the x, y, z axes suggested in the field formula.

[*Hint*: Find the potential inside and outside the sphere – the spherical harmonics provide a useful set of orthogonal functions]

Exercise 3. A conducting sphere with radius R carrying charge Q sits at the origin. The space outside the sphere above the $z = 0$ plane is a dielectric of constant κ_1. The space outside the sphere below the $z = 0$ plane is a dielectric of constant κ_2.

Find the potential everywhere and find the distributions of free and polarization charge in the system.
[*Hint*: Remember that the potential at every point on the conductor's surface must be the same.]

Exercise 4. The figure shows a cross-section of an infinite conducting cylindrical shell of radius R. Two thin strips of insulting material divide the cylinder into two segments, one of which is held at zero potential and the other at a potential V.



4.1 Show that the potential everywhere inside the cylinder is

$$\varphi(\rho, \phi) = \frac{\alpha V}{\pi} + \frac{2V}{\pi} \sum_{m=1}^{\infty} \frac{\sin m\alpha}{m} \left(\frac{\rho}{R}\right)^m \cos m\phi.$$

4.2 Using a computer, plot the potential as a function of angle, ϕ, in the case $\alpha = \pi/4$, for $\rho/R = 0.2, 0.4, 0.6, 0.8, 0.9, 0.95, 0.98, 0.999$.

4.3 Find the electric field inside the cylinder and show that the surface charge density induced on the inner surface of the cylinder is

$$\sigma(\phi) = \frac{\epsilon_0 V}{\pi R} \frac{\sin \alpha}{\cos \phi - \cos \alpha}.$$

Exercise 5. A point charge q, is placed a distance D from the center of a grounded conducting sphere of radius R, with the charge outside the sphere, $D > R$. We found the potential everywhere outside the sphere in class. Show that there is no tangential component of electric field at the sphere, and find the distribution of induced surface charge. Check that the total induced surface charge agrees with the image charge we considered in the notes.

Now suppose a point charge q, is placed a distance D from the center of a conducting sphere of radius R *carrying a charge* Q . The charge is outside the sphere, $D > R$.

5.1 Show that the force felt by the point charge is

$$\frac{q}{4\pi\epsilon_0}\left[\frac{Q+\frac{R}{D}q}{D^2}-\frac{\frac{R}{D}q}{\left(D-\frac{R^2}{D}\right)^2}\right]. \tag{1}$$

[*Hint:* You can use the image-charge result from the grounded sphere if you superpose it with another field to ensure that the sphere carries charge Q.]

5.2 Investigate the behavior of this force with D and observe that for small enough D the force can be attractive even if q and Q have the same sign. Explain physically what's happening here. In the case $q = Q$, show that the force on q is attractive for $D \lesssim 1.618\,R$ and repulsive otherwise.
[*Hint:* you may need to numerically find the roots of a polynomial.]

Exercise 6. A grounded conducting spherical shell of radius R has its center at the origin. Find the potential inside the sphere in the case that a point charge q is placed a distance $a < R$ from the origin.

Now suppose two point charges, $\pm q$, are located at positions $\pm a\hat{z}$ relative to the origin of a grounded conducting spherical shell of radius $R > a$.

6.1 Find an expression for the potential everywhere inside the sphere.

6.2 Consider the limit $a \to 0$, $q \to \infty$ with $p = 2qa$ held constant. What is the potential? What is the surface charge density on the conducting shell?

Exercise 7. The plane $z = 0$ is grounded except for a finite area S_0 which is held at a potential V.

7.1 Show that the potential away from the plane is

$$\varphi(\vec{r}) = \frac{V|z|}{2\pi} \int_{S_0} dS' \frac{1}{|\vec{r} - \vec{r}'|^3}. \tag{2}$$

[*Hint:* Obtain the (Dirichlet) Green's function for a system featuring an infinite plane, $G_D(\vec{r}, \vec{r}') = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{[(x-x')^2+(y-y')^2+(z-z')^2]^{1/2}} - \frac{1}{[(x-x')^2+(y-y')^2+(z+z')^2]^{1/2}}\right]$.
Hint: Hint: think about the method of images for a point charge next to a conducting plane]

7.2 In the case that the area S_0 is a circle of radius R centered at the origin, find the potential at points lying on the z axis.

Exercise 8. An infinite slab of dielectric lies between $z = a$ and $z = a + L$. A point charge, q, sits at the origin.

8.1 The most general solution of Laplace's equation relevant here is

$$\begin{aligned}
\varphi(\rho, \phi, z < a) &= \int_0^\infty dk\, A(k) J_0(k\rho)\, e^{kz} \\
\varphi(\rho, \phi, a < z < a + L) &= \int_0^\infty dk\, B(k) J_0(k\rho)\, e^{-kz} + \int_0^\infty dk\, C(k) J_0(k\rho)\, e^{kz} \\
\varphi(\rho, \phi, z > a + L) &= \int_0^\infty dk\, D(k) J_0(k\rho)\, e^{-kz}.
\end{aligned}$$

Explain why only J_0 features and why there is an integral over continuous k rather than a sum over discrete values of k. Explain the various factors of e^{kz}, e^{-kz}.

8.2 Given that the point charge provides a potential $\frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{\rho^2+z^2}}$, which has a Bessel function representation (see the first homework), apply suitable matching conditions to show that the potential for $z > a + L$ is

$$\varphi(z > a + L) = \frac{q(1-\gamma^2)}{4\pi\epsilon_0} \int_0^\infty dk\, \frac{J_0(k\rho)\, e^{-kz}}{1 - \gamma^2 e^{-2kL}} = \frac{q(1-\gamma^2)}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{\gamma^{2n}}{\sqrt{(z + 2nL)^2 + \rho^2}},$$

where $\gamma = (\kappa - 1)/(\kappa + 1)$ if the dielectric has constant κ.

8.3 Sketch the degree to which the point charge is screened at points along the z axis for $z > a + L$. Plot z in units of L, choosing a few values of κ.