Electromagnetism HW 7 – magnetostatics

1-4 due Mon 16th Nov, 5-8 due Mon 23th Nov

Exercise 1. A cylindrical conductor of radius a has a hole of radius b bored parallel to, and centered a distance d from the cylinder axis. The current density is uniform throughout the remaining metal of the conductor and is parallel to the cylinder axis. Use Ampere's law and the principle of superposition to find the magnetic field in the hole.

[*Hint:* Show that the magnetic field inside a cylindrical conductor with its axis pointing along the z-axis carrying a uniform current density of J is

$$\vec{B} = \frac{\mu_0 J}{2} \hat{z} \times (\vec{\rho} - \vec{d}),$$

if $\vec{d}$ is the position of the cylinder axis.]

Exercise 2. Show that each of the following vector potentials corresponds to a uniform $\vec{B}$-field pointing along the z-axis.

$$\begin{aligned}
\vec{A}^{(1)}(\vec{r}) &= xB_0\hat{y} \\
\vec{A}^{(2)}(\vec{r}) &= -\tfrac{1}{2}yB_0\hat{x} + \tfrac{1}{2}xB_0\hat{y} \\
\vec{A}^{(3)}(\vec{r}) &= -yB_0\hat{x} \\
\vec{A}^{(4)}(\vec{r}) &= \alpha yz\hat{x} + (\alpha xz + xB_0)\hat{y} + \alpha xy\hat{z} \\
\vec{A}^{(5)}(\vec{r}) &= \alpha\sin(\beta y)e^{\gamma z}\hat{x} + (\alpha\beta x\cos(\beta y)e^{\gamma z} + xB_0)\hat{y} + \alpha\gamma x\sin(\beta y)e^{\gamma z}\hat{z}
\end{aligned}$$

2.1: In each case $(2)-(5)$ find the gauge transform function $\Psi(\vec{r})$ in $\vec{A}^{(n)}(\vec{r}) = \vec{A}^{(1)}(\vec{r}) + \vec{\nabla}\Psi$.

2.2: In which cases $(1)-(5)$ does the Coulomb gauge condition hold ?

Exercise 3. A flat sheet of infinite extent lies in the xy plane carrying a uniform surface current, K_0, in the $\hat{x}$ direction. Show that the magnetic field is $\frac{1}{2}\mu_0 K_0$ in the $-\hat{y}$ direction above the sheet ($z > 0$) and $\frac{1}{2}\mu_0 K_0$ in the $+\hat{y}$ direction below the sheet ($z < 0$) by

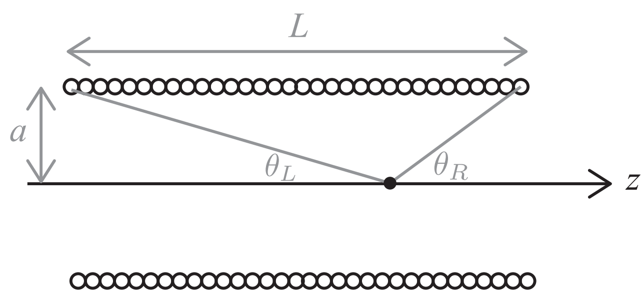
(a) considering an appropriate closed path in Ampere's law, and

(b) by using the Biot-Savart law.

Exercise 4. A cylindrical solenoid, of circular cross section with radius a, is of finite length, L, and features n turns per unit length. If the current carried is I, show that the magnetic field on the cylinder axis in the limit $nL \to \infty$ is

$$B_z = \frac{\mu_0 n I}{2}\big(\cos\theta_L + \cos\theta_R\big),$$

where the angles are defined in the figure.



[*Hint:* You might try superimposing the field from a number of circular coils before taking the limit.]

Exercise 5. In this exercise we'll consider the field from a pair of long parallel wires carrying equal current in opposite directions.

5.1: A single long straight wire carries current I along the z-axis. Using Ampere's law show that the field is $\vec{H} = \frac{I}{2\pi\rho}\hat{\phi}$. Away from the location of the current we can write $\vec{H} = -\vec{\nabla}\varphi_M$. Show that $\varphi_M(\rho, \phi) = -\frac{I}{2\pi}\phi$ gives the correct magnetic field.

5.2: Two wires lie parallel to the z-axis at positions $x = \pm d/2$, $y = 0$ carrying a current I in opposite directions. Show that in the limit of small d, the magnetic scalar potential is

$$\varphi_M \approx -\frac{I}{2\pi}\frac{d}{\rho}\sin\phi + \mathcal{O}\left(\frac{d}{\rho}\right)^2$$

[*Hint:* one way to get this is to use cartesian (x, y) coordinates and consider the expansion of $\tan(\phi + \epsilon)$ for small ϵ.]

5.3: Find the magnetic field from the pair of wires in the limit of small d and roughly sketch the field lines in the xy plane.

Exercise 6. Suppose the pair of wires in the previous exercise is surrounded by a circular cylinder of magnetic material of inner radius $a \gg d$, outer radius b, having permeability $\mu = \mu_r\mu_0$. Show that the field outside the cylinder is reduced by a factor

$$\frac{4\mu_r(b^2/a^2)}{(\mu_r + 1)^2(b^2/a^2) - (\mu_r - 1)^2}.$$

How much shielding do we get for a 2 mm thick steel cylinder ($\mu_r \sim 200$) of inner radius 1 cm ?

Exercise 7. A hollow sphere of internal radius a and external radius b has a uniform spontaneous magnetization, M. Show that there is zero magnetic field in the cavity ($r < a$) and that the external field ($r > b$) is the same as that of a dipole moment $\vec{m} = \frac{4\pi}{3}(b^3 - a^3)\vec{M}$.

Exercise 8. We've previously considered 'electrostatic' situations in which point electric charges remain in the same location and found that Coulomb's law describes the forces between them. However we know from special relativity that the physics must be independent of the frame from which we view the system. If we viewed the electrostatic system from a moving frame, the charges would appear to be in motion. This problem applies the frame transformation properties of special relativity to explore what happens (we'll consider this in more detail next semester).

Suppose frame 2 moves with velocity $\vec{u} = u\hat{x}$ with respect to frame 1, then the space-time co-ordinates of a point in frame 2 are related to those in frame 1 by,

$$x^{(2)} = \gamma\big(x^{(1)} - ut^{(1)}\big);\;\; y^{(2)} = y^{(1)};\;\; z^{(2)} = z^{(1)};\;\; t^{(2)} = \gamma\big(t^{(1)} - \tfrac{u}{c^2}x^{(1)}\big), \tag{1}$$

where $\gamma = \big(1 - \frac{u^2}{c^2}\big)^{1/2}$. The momentum and energy of a particle of rest mass, m, moving with velocity $\vec{v}$ are $\vec{p} = \gamma m\vec{v}$, $E = \gamma mc^2$ and they transform according to

$$p_x^{(2)} = \gamma\big(p_x^{(1)} - \tfrac{u}{c^2}E^{(1)}\big);\;\; p_y^{(2)} = p_y^{(1)};\;\; p_z^{(2)} = p_z^{(1)};\;\; E^{(2)} = \gamma\big(E^{(1)} - up_x^{(1)}\big). \tag{2}$$

8.1: Suppose a particle is at rest in frame 2 and feels a force $\vec{F}^{(2)}$. Show that the force it feels in frame 1 is

$$F_x^{(1)} = F_x^{(2)};\;\; F_y^{(1)} = \tfrac{1}{\gamma}F_y^{(2)};\;\; F_z^{(1)} = \tfrac{1}{\gamma}F_z^{(2)}. \tag{3}$$

[*Hint*: $F_i^{(1)} = \frac{dp_i^{(1)}}{dt^{(1)}} = \frac{dt^{(2)}}{dt^{(1)}}\frac{dp_i^{(1)}}{dt^{(2)}}$]

8.2: Suppose a charge q_a is at rest at the origin of frame 2, and a charge q_b is at rest at position $(x^{(2)}, y^{(2)}, 0)$. Write down the components of the Coulomb force on q_b due to q_a in frame 2. Find the force on q_b as observed in frame 1, measured at time $t^{(1)} = 0$, and show that it can be written

$$q_b\left[\gamma\frac{q_a}{4\pi\epsilon_0}\frac{\vec{r}^{(1)}}{\big(\gamma^2x^{(1)2} + y^{(1)2}\big)^{3/2}} + \vec{u}\times\left\{\gamma\frac{q_a u}{4\pi\epsilon_0 c^2}\frac{y^{(1)}\hat{z}}{\big(\gamma^2x^{(1)2} + y^{(1)2}\big)^{3/2}}\right\}\right], \tag{4}$$

which we might write as $q_b\left[\vec{E} + \vec{u}\times\vec{B}\right]$. The first term in this expression can indeed be identified with the Coulomb field from charge q_a, as can be seen be taking the limit $u \to 0$. The second term features the *magnetic field*, and we see that special relativity insists that if electric fields exist, so must magnetic fields, and they must be related to each other in a very particular way.