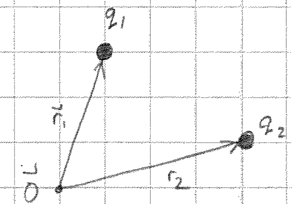
COULOMB'S LAW OF ELECTROSTATICS

Empirically we find that the force between two small charged bodies of charge q_1 & q_2 seperated by a distance $|\vec{r}_1 - \vec{r}_2|$ is



(force on q_1) $$\vec{F}_{12} = k \cdot q_1 q_2 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} = k \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^2} \cdot \hat{r}_{12}$$

in the SI units system where distances are measured in meters & charge is measured in Coulombs, $k = \frac{1}{4\pi\varepsilon_0}$

where $\varepsilon_0 \simeq 8.854 \times 10^{-2}$ F/m is the "permittivity of free space"

We can define a quantity which describes the effect of charge q_2 at the position $\vec{r}_1$ irrespective of what charge is placed at $\vec{r}_1$. The "electric field" is defined via the force felt by a test charge at $\vec{r}_1$ in the limit of an infinitesimally small test charge:

$$\vec{E}(\vec{r}) = \lim_{q \to 0} \left[\frac{\vec{F}(\vec{r})}{q} \right]$$

hence the electric field at position $\vec{r}$ due to a point charge, q, at $\vec{r}\,'$ is

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} q \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3}$$

Empirically we find that the force from a set of charges is the linear superposition of the force from each charge - similarly for the electric field

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_i q_i \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|^3}$$

In many cases it is convenient to treat the distribution of charge over space as being continuous & described by a charge density, $\rho(\vec{r}\,')$, and in this case

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3\vec{r}\,' \rho(\vec{r}\,') \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3}$$

note that a system of point charges can be described using delta functions

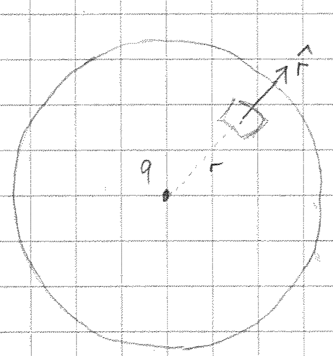
$$\rho(\vec{r}\,') = \sum_i q_i \, \delta(\vec{r}\,' - \vec{r}_i)$$

COULOMB & GAUSS

Consider a single point charge, q, located at the origin of a co-ordinate system, then

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r}$$

now consider integrating the "flux" of the electric field, $\vec{E}\cdot d\vec{S}$, over the surface of a sphere of radius, r :

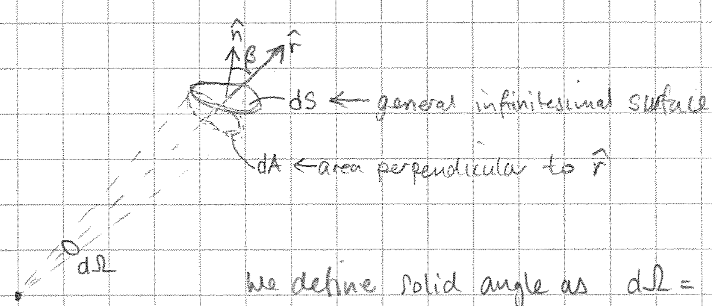


$$d\vec{S} = \hat{r}\,.\,r^2 \sin\theta\, d\theta\, d\phi$$

$$\vec{E}(\vec{r})\cdot d\vec{S} = \frac{q}{4\pi\varepsilon_0}\,.\,\sin\theta\, d\theta\, d\phi$$

$$\int \vec{E}(\vec{r})\cdot d\vec{S} = \frac{q}{4\pi\varepsilon_0}\,.\,4\pi = q/\varepsilon_0$$ independent of $|\vec{r}|$

In fact this result does not depend upon the shape of the surface enclosing the charge

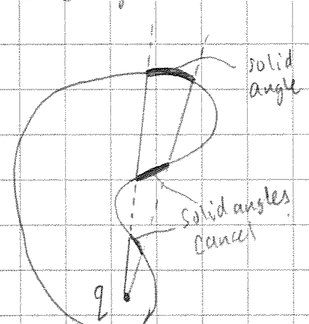


$$dA = dS\cos\beta = dS\,\hat{n}\cdot\hat{r}$$

we define solid angle as $d\Omega = \dfrac{dA}{r^2} = dS\,\dfrac{\hat{n}\cdot\hat{r}}{r^2} = \dfrac{\hat{r}\cdot d\vec{S}}{r^2}$

then the "flux" of the electric field through dS is $\vec{E}\cdot d\vec{S} = \dfrac{q}{4\pi\varepsilon_0}\, d\Omega$

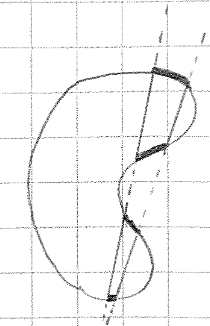
& if the charge is surrounded by a surface S, then the integral $\int d\Omega = 4\pi$



$$\Rightarrow \int_S \vec{E}\cdot d\vec{S} = q/\varepsilon_0$$

it the charge lies *outside* the surface, $\int d\Omega = 0$

$$\Rightarrow \int_S \vec{E}\cdot d\vec{S} = 0$$



the extension to a continuous charge distribution reads

$$\boxed{\oint_S \vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon_0}\int_V d^3\vec{r}\,\rho(\vec{r})}$$

where V is the volume enclosed by the surface S.

"Gauss's Law"

$\int_V d^3\vec{r}\,\rho(\vec{r}) = Q_{TOT}$ ← total charge enclosed in V

Use of the divergence theorem $\oint_S \vec{A}\cdot d\vec{S} = \int_V dV\ \vec{\nabla}\cdot\vec{A}$

we may rewrite this as

$$\int d^3\vec{r}\,(\vec{\nabla}\cdot\vec{E})(\vec{r}) = \frac{1}{\varepsilon_0}\int d^3r\,\rho(\vec{r})$$

$$\Rightarrow \int d^3\vec{r}\left(\vec{\nabla}\cdot\vec{E} - \rho/\varepsilon_0\right) = 0$$

& since this must hold true for any volume choice, the integrand must be zero:

$$\boxed{\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0}$$

"differential form of Gauss's Law"

another equation involving $\vec{E}$ follows from Coulomb's law:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3\vec{r}\,' \rho(\vec{r}\,') \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

note that $\quad \dfrac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} = -\vec{\nabla}_{\vec{r}} \left(\dfrac{1}{|\vec{r}-\vec{r}\,'|} \right)$

so we may write

$$\vec{E}(\vec{r}) = -\frac{1}{4\pi\epsilon_0} \vec{\nabla}_{\vec{r}} \int d^3\vec{r}\,' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

so that $\vec{E}(\vec{r})$ is the gradient of a scalar function $\vec{E} = -\vec{\nabla}\phi$, $\phi(\vec{r}) = \dfrac{1}{4\pi\epsilon_0} \displaystyle\int d^3\vec{r}\,' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$

& it follows that $\boxed{\vec{\nabla}\times\vec{E} = 0}$

Application of Stokes' theorem $\left(\oint_C d\vec{l}\cdot\vec{A} = \int_S d\vec{S}\cdot(\vec{\nabla}\times\vec{A}) \right)$ to this result

suggests that $\oint_C \vec{E}.d\vec{l} = 0$, i.e the integral of the electric field around <u>any</u> closed path is zero.

The electrostatic force is hence "conservative" - we already know that we can define the corresponding potential, $\vec{E} = -\vec{\nabla}\phi$

Since we know that $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$, the potential must satisfy $\boxed{\nabla^2\phi = -\rho(\vec{r})/\epsilon_0}$

"Poisson's equation"

We can show that the potential, ϕ, is simply related to the energy stored in a system of charges, the "electrostatic potential energy"

recall the work-energy theorem $K_f = K_i + W$

where W is the work done by some force, here the electrostatic force $\vec{F} = q\vec{E}$

suppose we move a charge q in a pre-existing electric field $\vec{E}(\vec{r})$, then the work done by the electric field moving from $\vec{r}_i$ to $\vec{r}_f$ is

$$W = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}.d\vec{l} = q\int_{\vec{r}_i}^{\vec{r}_f} \vec{E}.d\vec{l} = q\int_{\vec{r}_i}^{\vec{r}_f} (-\vec{\nabla}\phi).d\vec{l} = -q\Big[\phi\Big]_{\vec{r}_i}^{\vec{r}_f} = -q\,\phi(\vec{r}_f) + q\,\phi(\vec{r}_i)$$

but since we express energy conservation as $K_f + U_f = K_i + U_i \Rightarrow W = U_i - U_f$

& thus $\underline{U = q\,\phi(\vec{r})}$

Let's consider the potential energy of two point charges: $q_1 @ \vec{r}_1$ and $q_2 @ \vec{r}_2$

$$U_{12} = q_2 \cdot \phi_1(\vec{r}_2) = q_2 \cdot \frac{1}{4\pi\epsilon_0}\frac{q_2}{|\vec{r}_2 - \vec{r}_1|} = \underline{\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|}}$$

& now the potential is $\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q_1}{|\vec{r}-\vec{r}_1|} + \frac{1}{4\pi\epsilon_0}\frac{q_2}{|\vec{r}-\vec{r}_2|}$

suppose we bring in a third charge, $q_3 @ \vec{r}_3$:

now the energy of q_3 in the field of 1 & 2 is $U_{13} + U_{23} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_3}{|\vec{r}_1 - \vec{r}_3|} + \frac{1}{4\pi\epsilon_0}\frac{q_2 q_3}{|\vec{r}_2 - \vec{r}_3|}$

& the total potential energy is $U = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|} + \frac{1}{4\pi\epsilon_0}\frac{q_1 q_3}{|\vec{r}_1 - \vec{r}_3|} + \frac{1}{4\pi\epsilon_0}\frac{q_2 q_3}{|\vec{r}_2 - \vec{r}_3|}$

bringing in further charges gives

$$\underline{U = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\sum_{j<i}\frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}}$$

which we can rewrite in a more symmetric way as

(D) $$U = \frac{1}{4\pi\varepsilon_0} \cdot \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|} = \frac{1}{2} \sum_i q_i \phi_i$$

potential from all but q_i

we propose that the appropriate form for continuous distributions is

(C) $$U = \frac{1}{4\pi\varepsilon_0} \cdot \frac{1}{2} \int d^3\vec{r}\, d^3\vec{r}\,' \frac{\rho(\vec{r})\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} = \frac{1}{2} \int d^3\vec{r}\, \rho(\vec{r}) \phi(\vec{r})$$

but we need to be careful with this because (C) contains "self-energy" contributions not present in (D):

e.g. $\rho(\vec{r}) = \sum_i q_i \delta(\vec{r} - \vec{r}_i)$ $\rightsquigarrow$ a set of discrete charges

$$U = \frac{1}{4\pi\varepsilon_0} \frac{1}{2} \int d^3\vec{r}\, d^3\vec{r}\,' \frac{1}{|\vec{r} - \vec{r}\,'|} \sum_{i,j} q_i q_j \delta(\vec{r} - \vec{r}_i) \delta(\vec{r}\,' - \vec{r}_j)$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{1}{2} \int d^3\vec{r} \sum_{i,j} \frac{q_i q_j}{|\vec{r} - \vec{r}_j|} \delta(\vec{r} - \vec{r}_i) = \underline{\frac{1}{4\pi\varepsilon_0} \frac{1}{2} \sum_{ij} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}}$$

but this contains terms where $i = j \longrightarrow \frac{q_i^2}{|\vec{r}_i - \vec{r}_i|} \longrightarrow \infty$!

→ this is the infinite energy needed to "assemble" a point charge

→ we're exposing the internal inconsistency of the "point" charge concept.

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another useful way of viewing U comes from eliminating $\rho(\vec{r}), \phi(\vec{r})$ in favour of $\vec{E}(\vec{r})$:

$U = \frac{1}{2} \int d^3\vec{r}\, \rho(\vec{r}) \phi(\vec{r})$ but $\rho(\vec{r}) = \varepsilon_0 \vec{\nabla} \cdot \vec{E} = -\varepsilon_0 \nabla^2 \phi$ so $U = -\frac{\varepsilon_0}{2} \int d^3\vec{r}\, \phi(\vec{r}) \nabla^2 \phi(\vec{r})$

integrating by parts: $u = \phi$ $v' = \nabla^2 \phi$
$u' = \vec{\nabla}\phi$ $v = \vec{\nabla}\phi$

$$U = -\frac{\varepsilon_0}{2} \left[\phi \vec{\nabla} \phi\right]_{\substack{\text{surface} \\ \text{at } \infty}} + \frac{\varepsilon_0}{2} \int d^3\vec{r}\, |\vec{\nabla}\phi|^2$$

$U = \frac{\varepsilon_0}{2} \int d^3\vec{r}\, |\vec{E}(\vec{r})|^2$ & we can associate $\frac{1}{2} \varepsilon_0 |\vec{E}(\vec{r})|^2$ with the "energy density" – ie the energy stored per unit volume in the electric field.

we can define the "potential energy of interaction" to exclude self-energies:

consider separating a charge distribution $\rho(\vec{r}) = \rho_1(\vec{r}) + \rho_2(\vec{r})$

then
$$U = \frac{1}{4\pi\epsilon_0}\frac{1}{2}\int d^3\vec{r}\, d^3\vec{r}\,'\, \frac{\rho_1(\vec{r})\rho_1(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} + \frac{1}{4\pi\epsilon_0}\frac{1}{2}\int d^3\vec{r}\, d^3\vec{r}\,'\, \frac{\rho_2(\vec{r})\rho_2(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

$$+ \frac{1}{4\pi\epsilon_0}\frac{1}{2}\cdot 2\cdot \int d^3\vec{r}\, d^3\vec{r}\,'\, \frac{\rho_1(\vec{r})\,\rho_2(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

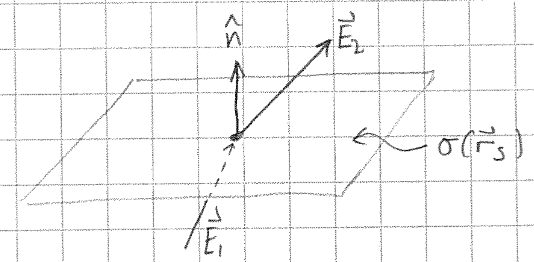
where the first two terms are self-energies and the third term is the potential energy of interaction between two independent charge distributions $\rho_1(\vec{r}), \rho_2(\vec{r})$

$$U_{int} = \frac{1}{4\pi\epsilon_0}\int d^3\vec{r}\, d^3\vec{r}\,'\, \frac{\rho_1(\vec{r})\rho_2(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} = \int d^3\vec{r}\, \rho_1(\vec{r})\,\phi_2(\vec{r})$$

NB no factor of $\frac{1}{2}$!

SURFACE CHARGE DISTRIBUTIONS

A situation we'll encounter later is when charge is distributed on a surface - the behavior of the $\vec{E}$-field an the potential in the immediate vicinity of the surface can be found easily

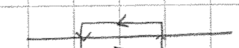


a pillbox of infinitesimal height having its top above the surface & its bottom below provides a Gaussian surface such that

$$\oint_S \vec{E}\cdot d\vec{s} = \vec{E}_2\cdot\hat{n}\,A + \vec{E}_1\cdot(-\hat{n})\,A = \frac{Q_{TOT}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

$$\Rightarrow \boxed{\hat{n}\cdot(\vec{E}_2 - \vec{E}_1) = \sigma/\varepsilon_0}$$

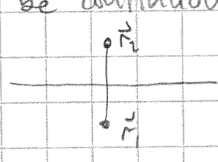
the perpendicular component of the $\vec{E}$-field is discontinuous

integrating $\vec{E}$ around a path  σ

will yield the fact that the parallel components of the $\vec{E}$-field are continuous

$$\boxed{\hat{n}\times(\vec{E}_2 - \vec{E}_1) = 0}$$

the potential can be shown to be continuous by integrating $\vec{E}$ along an open path going through the surface



$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{E}\cdot d\vec{\ell} = \int_{\vec{r}_1}^{\vec{r}_2} -\vec{\nabla}\phi\cdot d\vec{\ell} = -\left(\phi(\vec{r}_2) - \phi(\vec{r}_1)\right)$$

but if we choose $\vec{r}_1$ just below & $\vec{r}_2$ just above the surface so that $|d\vec{\ell}| \to 0$, we would only have a discontinuity in ϕ if $\vec{E}$ diverged - but we know that $\vec{E}$ does not divergence

while ϕ is continuous, its derivative in the direction across the surface is discontinuous.

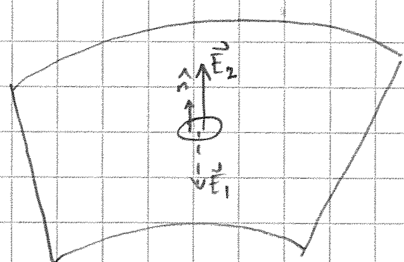
[we'll see later that ϕ can be discontinuous for a surface distribution of dipoles]

We might wonder how much force the various areas of surface charge density apply to each other.

The usual approach is to say an infinitesimal area, dA, of surface charge density, σ will feel a force $d\vec{F} = (\sigma dS)\vec{E}$ where $\vec{E}$ is the field at the location of σ.

but it's not obvious what $\vec{E}$ should be since $\vec{E}$ is discontinuous at σ.

Let's be more careful. Consider a large surface carrying surface charge density $\sigma(\vec{r}_s)$. Now consider an infinitesimal disk at position $\vec{r}_s$ – that disk cannot apply a force to itself, so let's remove that bit of surface charge & consider the fields due to the rest of the surface



$$\vec{E}_1 = \vec{E}_{rest} + \vec{E}^{(1)}_{disk} = \vec{E}_{rest} - \sigma/2\epsilon_0\,\hat{n}$$

$$\vec{E}_2 = \vec{E}_{rest} + \vec{E}^{(2)}_{disk} = \vec{E}_{rest} + \sigma/2\epsilon_0\,\hat{n}$$

(e.g. see exercise 1.2 in the limit that we're very close to the disk)

then we observe that $\vec{E}_2 - \vec{E}_1 = \frac{\sigma}{\epsilon_0}\hat{n}$ as we expect,

but now we can say that the force on the infinitesimal disk of charge density at $\vec{r}_s$ is

$$d\vec{F} = (\sigma dS)\vec{E}_{rest} = (\sigma dA)\,\frac{1}{2}(\vec{E}_1 + \vec{E}_2)$$

$\Rightarrow$ the force density $\underline{\dfrac{d\vec{F}(\vec{r}_s)}{dS} = \dfrac{1}{2}\sigma(\vec{r}_s)\left(\vec{E}_1(\vec{r}_s) + \vec{E}_2(\vec{r}_s)\right)}$

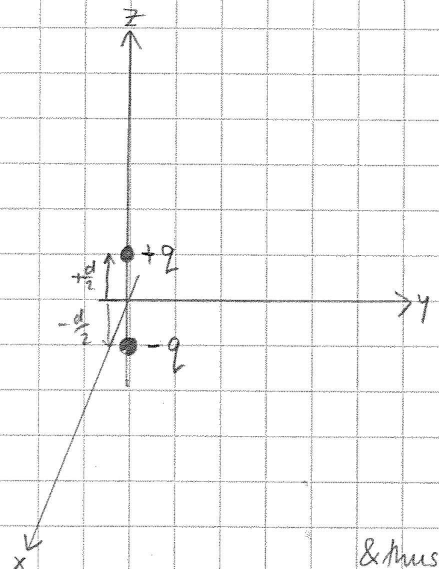
so it is the <u>average</u> field which appears.

Note that if there are other sources of electric field present:

$$\left.\begin{aligned}\vec{E}_1 &= \vec{E}_{ext} + \vec{E}_{rest} - \sigma/2\epsilon_0 \\ \vec{E}_2 &= \vec{E}_{ext} + \vec{E}_{rest} + \sigma/2\epsilon_0\end{aligned}\right\}$$ & the force $d\vec{F} = (\sigma dS)(\vec{E}_{ext} + \vec{E}_{rest}) = (\sigma dS)\frac{1}{2}(\vec{E}_1 + \vec{E}_2)$ still.

AN ELECTRIC DIPOLE

consider two electric charges of opposite sign but equal magnitude separated by a distance d



then the potential at any point $\vec{r}$ is given by

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{(-q)}{|\vec{r} - \frac{d}{2}\hat{z}|} + \frac{1}{4\pi\varepsilon_0}\frac{(+q)}{|\vec{r} + \frac{d}{2}\hat{z}|}$$

$$\frac{1}{|\vec{r} \pm \frac{d}{2}\hat{z}|} = \left(x^2 + y^2 + (z \pm \tfrac{d}{2})^2\right)^{-1/2}$$

$$\simeq \frac{1}{r}\left(1 \pm \frac{zd}{r^2}\right)^{-1/2} \simeq \frac{1}{r} \mp \frac{zd}{2r^3} + \dots$$

& thus at distances far from the dipole ($r >> d$),

$$\phi \rightarrow \frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{r} - \frac{zd}{2r^3} + \dots + \frac{1}{r} - \frac{zd}{2r^3} + \dots\right]$$

$$= \frac{qd}{4\pi\varepsilon_0} \cdot \frac{z}{r^3}$$

we call the product, $q \cdot d$, the "electric dipole moment" $p = qd$

$$= \frac{p}{4\pi\varepsilon_0} \, z/r^3$$

for a pair oriented in a general direction we can define the electric dipole moment vector $\vec{p} = q\vec{d}$ where $\vec{d}$ points from the $-q$ charge to the $+q$ charge

$$\& \quad \phi(r \to \infty) \rightarrow \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3}$$

note that this falls like $1/r^2$ with distance, more rapid than the $\frac{1}{r}$ of a point charge the electric field can be obtained as the gradient, $\vec{E} = -\vec{\nabla}\phi$

far from the dipole

$$\phi(\vec{r}) = \frac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 r^3}$$

using identities:

$$\vec{\nabla}\left(\frac{\vec{p}\cdot\vec{r}}{r^3}\right) = \frac{1}{r^3}\vec{\nabla}(\vec{p}\cdot\vec{r}) + \vec{p}\cdot\vec{r}\,\vec{\nabla}\left(\frac{1}{r^3}\right) = \frac{1}{r^3}\left((\vec{p}\cdot\vec{\nabla})\vec{r} + (\vec{r}\cdot\vec{\nabla})\vec{p} + \vec{p}\times(\vec{\nabla}\times\vec{r}) + \vec{r}\times(\vec{\nabla}\times\vec{p})\right) + \vec{p}\cdot\vec{r}\,\frac{-3\hat{r}}{r^4}$$

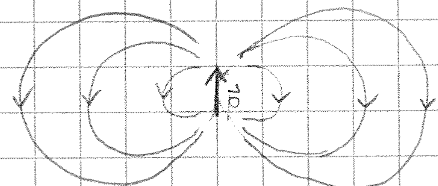
$$= \vec{p}/r^3 - 3\,\vec{p}\cdot\vec{r}\;\hat{r}/r^4$$

if the z-axis is chosen to point along $\vec{p}$: $\phi(r,\theta,\phi) = \dfrac{p\,r\cos\theta}{4\pi\epsilon_0 r^3} = \dfrac{p}{4\pi\epsilon_0}\dfrac{\cos\theta}{r^2}$

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

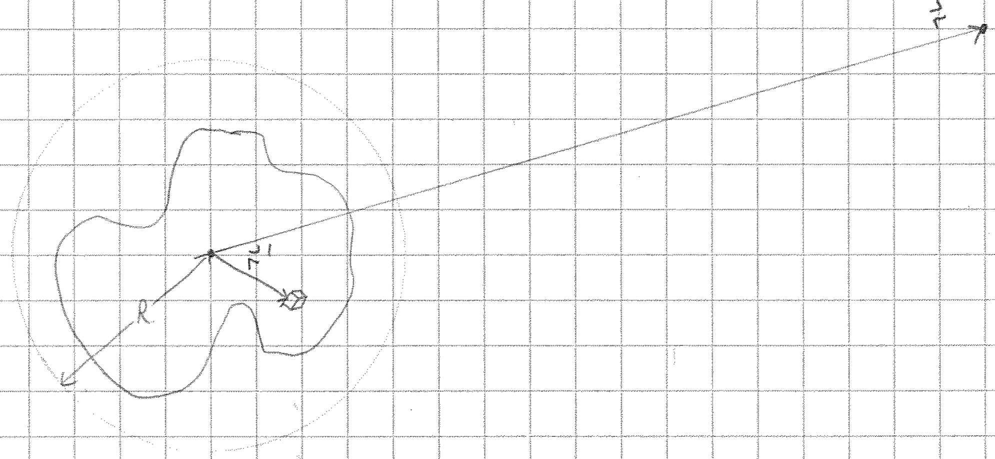
$$\vec{E}(r,\theta,\phi) = -\frac{p}{4\pi\epsilon_0}\left(\hat{r}\,\frac{-2\cos\theta}{r^3} + \hat{\theta}\,\frac{1}{r}\,\frac{-\sin\theta}{r^2}\right) = \frac{p}{4\pi\epsilon_0}\frac{1}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{3\hat{r}(\vec{p}\cdot\hat{r}) - \vec{p}}{r^3}\right)$$



THE ELECTRIC MULTIPOLE EXPANSION

Suppose we have a distribution of charge, $\rho(\vec{r}\,')$, which sits entirely within some region, say a sphere of radius R



We can develop a systematic approximation for the potential and $\vec{E}$-field valid at distances $r \gg R$, called the "electric multipole expansion".

the exact potential is $\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3\vec{r}\,' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$

but since $|\vec{r}\,'|<R$ & $r \gg R$ we can approximate $\frac{1}{|\vec{r}-\vec{r}\,'|}$ by a few terms of its Taylor series

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r} - \vec{r}\,'\cdot\vec{\nabla}\frac{1}{r} + \frac{1}{2}\left(\vec{r}\,'\cdot\vec{\nabla}\right)^2\frac{1}{r} + \dots$$

& thus
$$\phi(\vec{r}) \simeq \frac{1}{4\pi\varepsilon_0}\Bigg[\left(\int d^3\vec{r}\,'\rho(\vec{r}\,')\right)\frac{1}{r} - \left(\int d^3\vec{r}\,'\rho(\vec{r}\,')\vec{r}\,'\right)\cdot\vec{\nabla}\frac{1}{r} + \left(\frac{1}{2}\int d^3\vec{r}\,'\rho(\vec{r}\,')r_i'r_j'\right)\nabla_i\nabla_j\frac{1}{r} + \dots\Bigg]$$

the object $\int d^3\vec{r}\,'\rho(\vec{r}\,')$ is the total charge, Q

the object $\int d^3\vec{r}\,'\rho(\vec{r}\,')\vec{r}\,'$ is the charge weighted by position, we'll call it the <u>electric dipole moment</u> $\vec{p}$

$\left(\text{e.g. } \rho(\vec{r}\,') = \sum_n q_n\,\delta(\vec{r}\,'-\vec{r}_n) \Rightarrow \vec{p} = \sum_n q_n\vec{r}_n\right)$

the object $\frac{1}{2}\int d^3\vec{r}\,'\rho(\vec{r}\,')r_i'r_j'$ we call the (cartesian) <u>electric quadrupole moment</u>, Q_{ij} (or tensor)

$$\Rightarrow \phi(\vec{r}) \simeq \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r} + \frac{\vec{p}\cdot\vec{r}}{r^3} + Q_{ij}\frac{3r_ir_j - r^2\delta_{ij}}{r^5} + \dots\right]$$

an aside on electric fields & the electric dipole moment:

suppose $\vec{E}(\vec{r})$ is the electric field produced by a charge distribution $\rho(\vec{r})$, some of which lies inside a sphere of radius R, and some of which lies outside,

then $$\int_{V_{sph}} d^3\vec{r}\, \vec{E}(\vec{r}) = \int_{V_{sph}} d^3\vec{r} \cdot \frac{1}{4\pi\varepsilon_0} \int d^3\vec{s} \quad \rho(\vec{s}) \frac{(\vec{r}-\vec{s})}{|\vec{r}-\vec{s}|^3}$$

$$= \int d^3\vec{s}\, \frac{-\rho(\vec{s})}{4\pi\varepsilon_0} \cdot \int_{V_{sph}} d^3\vec{r}\, \frac{\vec{s}-\vec{r}}{|\vec{s}-\vec{r}|^3}$$

but $\frac{1}{4\pi\varepsilon_0} \int_{V_{sph}} d^3\vec{r}\, \frac{\vec{s}-\vec{r}}{|\vec{s}-\vec{r}|^3}$

is the $\vec{E}$-field at position $\vec{s}$ for a uniform spherical volume of charge (with $\rho(\vec{r}) = 1$)
$\Rightarrow$ total charge $Q = \frac{4}{3}\pi R^3 = V$

We showed in the homework that the field from a uniform sphere of charge is

$$\vec{E}(s<R) = \hat{s}\, \frac{Q}{4\pi\varepsilon_0} \frac{s}{R^3} = \vec{s}/3\varepsilon_0$$

$$\vec{E}(s>R) = \hat{s}\, \frac{Q}{4\pi\varepsilon_0} \frac{1}{s^2} = \frac{V}{4\pi\varepsilon_0} \frac{\vec{s}}{s^3}$$

& it follows that $$\int_{V_{sph}} d^3\vec{r}\, \vec{E}(\vec{r}) = \int_{s<R} d^3\vec{s} \left[-\rho(\vec{s})\right]\left[\vec{s}/3\varepsilon_0\right] + \int_{s>R} d^3\vec{s} \left[-\rho(\vec{s})\right]\left[\frac{V}{4\pi\varepsilon_0} \frac{\vec{s}}{s^3}\right]$$

$$= -\frac{1}{3\varepsilon_0} \int_{s<R} d^3\vec{s}\, \rho(\vec{s})\, \vec{s} - \frac{V}{4\pi\varepsilon_0} \int_{s>R} d^3\vec{s}\, \rho(\vec{s})\, \frac{\vec{s}}{s^3}$$

& notice that if all the charge lies inside the spherical volume we have

$$\underline{\int d^3\vec{s}\, \rho(\vec{s})\, \vec{s} = \vec{p} = -3\varepsilon_0 \int_{V_{sph}} d^3\vec{r}\, \vec{E}(\vec{r})}$$

also notice that if none of the charge lies inside the spherical volume,

$$\underline{+\frac{1}{4\pi\varepsilon_0} \int d^3\vec{s}\, \rho(\vec{s})\, \frac{(\vec{0}-\vec{s})}{|\vec{0}-\vec{s}|^3} = \vec{E}(\vec{0}) = \frac{1}{V} \int_{V_{sph}} d^3\vec{r}\, \vec{E}(\vec{r})}$$

so the electric field at the center of a spherical volume containing no charge is the average of the electric field over the volume

THE "POINT" ELECTRIC DIPOLE

we can define a useful object which is a dipole "at a point":

Start with charge $(-q/s)$ located at $\vec{r}_0$ & charge $(+q/s)$ located at $\vec{r}_0 + s\vec{d}$

then the potential at $\vec{r}$ is $\phi_s(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q/s}{|\vec{r}-\vec{r}_0-s\vec{d}|} + \frac{(-q/s)}{|\vec{r}-\vec{r}_0|}\right]$

the "point" electric dipole is defined as the limit $s\to 0$, keeping the moment $\vec{p} = (\frac{q}{s})(s\vec{d}) = q\vec{d}$ constant

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\lim_{s\to 0}\left[\frac{q/s}{|\vec{r}-\vec{r}_0-s\vec{d}|} - \frac{q/s}{|\vec{r}-\vec{r}_0|}\right]$$

$$\frac{q/s}{|\vec{r}-\vec{r}_0-s\vec{d}|} = \frac{q/s}{|\vec{r}-\vec{r}_0|} - \left(\frac{q}{s}\right)s\vec{d}\cdot\vec{\nabla}\left[\frac{1}{|\vec{r}-\vec{r}_0|}\right] + O(s)$$

$$\Rightarrow \underline{\phi(\vec{r}) = -\frac{1}{4\pi\varepsilon_0}\vec{p}\cdot\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}_0|}}$$

& this should be valid for all $\vec{r}$, not just far from $\vec{r}_0$.

note that the charge distribution that has provided this field is quite singular

using Poisson: $\rho(\vec{r}) = -\varepsilon_0\nabla^2\phi = \frac{1}{4\pi}\vec{p}\cdot\vec{\nabla}\underbrace{\nabla^2\frac{1}{|\vec{r}-\vec{r}_0|}}_{-4\pi\delta(\vec{r}-\vec{r}_0)} = -\vec{p}\cdot\vec{\nabla}\delta(\vec{r}-\vec{r}_0)$

directly from the limit $\rho(\vec{r}) = \lim_{s\to 0}\left[\left(\frac{+q}{s}\right)\delta(\vec{r}-\vec{r}_0-s\vec{d}) + \left(\frac{-q}{s}\right)\delta(\vec{r}-\vec{r}_0)\right]$

$$= \lim_{s\to 0}\left[\cancel{\frac{q}{s}\delta(\vec{r}-\vec{r}_0)} + \frac{q}{s}\left(-s\vec{d}\cdot\vec{\nabla}\delta(\vec{r}-\vec{r}_0)\right) + O(s) - \cancel{\frac{q}{s}\delta(\vec{r}-\vec{r}_0)}\right] = -\vec{p}\cdot\vec{\nabla}\delta(\vec{r}-\vec{r}_0)$$

What about the electric field?

Consider $\vec{r}_0 = \vec{0}$ for simplicity, then $\phi(\vec{r}) = -\frac{1}{4\pi\varepsilon_0}\vec{p}\cdot\vec{\nabla}\frac{1}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}$

& $\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{3\hat{r}(\vec{p}\cdot\hat{r}) - \vec{p}}{r^3}\right]$ as we found before, but now this should be good for all $\vec{r}$

if we integrate over a sphere centered on the point dipole we should get $\int_{V_{sph}} d^3\vec{r}\,\vec{E}(\vec{r}) = -\vec{p}/3\varepsilon_0$

but we actually get $\int_{V_{sph}} d^3\vec{r}\,\vec{E}(\vec{r}) = 0$ [CHECK IT YOURSELF]

$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{3\hat{r}(\vec{p}\cdot\hat{r}) - \vec{p}}{r^3} - \frac{4\pi}{3}\vec{p}\,\delta(\vec{r})\right]$

We'd expect there to be some energy associated with placing a dipole in an existing $\vec{E}$-field:

the potential energy of interaction is $U_{int} = \int d^3\vec{r}\, \rho_{dip}(\vec{r})\, \Phi_{ext}(\vec{r})$

& for a point electric dipole we already found that $\rho(\vec{r}) = -\vec{p}\cdot\vec{\nabla}\,\delta(\vec{r}-\vec{r}_0)$

& thus $U_{int} = -\vec{p}\cdot\int d^3\vec{r}\, \Phi_{ext}(\vec{r})\, \vec{\nabla}\,\delta(\vec{r}-\vec{r}_0)$

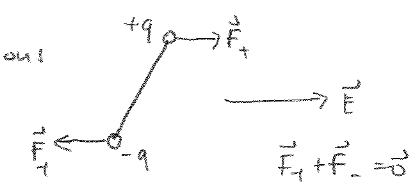
& integrating by parts $U_{int} = \vec{p}\cdot\int d^3\vec{r}\, \delta(\vec{r}-\vec{r}_0)\, \vec{\nabla}\Phi_{ext} = -\vec{p}\cdot\vec{E}(\vec{r}_0)$

$\underline{U_{int} = -\vec{p}\cdot\vec{E}(\vec{r}_0)}$ for a point dipole at $\vec{r}_0$

- the force applied to the dipole can be obtained as the negative gradient of the potential energy

$$\vec{F} = -\vec{\nabla} U_{int} = \vec{\nabla}(\vec{p}\cdot\vec{E}) = (\vec{p}\cdot\vec{\nabla})\vec{E}$$

so a non-uniform $\vec{E}$-field is required as should be obvious



- what about a torque?

Suppose the dipole is rotated rigidly by an infinitesimal amount $\delta\vec{\alpha}$ about its center, then the dipole moment changes by $\delta\vec{p} = \delta\vec{\alpha}\times\vec{p}$

& the interaction energy changes by $\delta U_{int} = -\delta\vec{p}\cdot\vec{E} = -\delta\vec{\alpha}\times\vec{p}\cdot\vec{E} = -(\vec{p}\times\vec{E})\cdot\delta\vec{\alpha}$

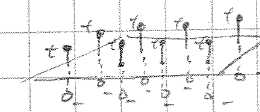
but a rotation by $\delta\vec{\alpha}$ when a torque $\vec{\tau}$ is applied leads to an energy change $-\vec{\tau}\cdot\delta\vec{\alpha}$

& thus $\underline{\vec{\tau} = \vec{p}\times\vec{E}}$

* ~~could be skipped~~

ELECTRIC DIPOLE LAYERS

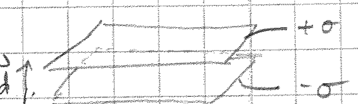
Consider a distribution of point electric dipoles confined to a surface



each dipole has a moment $\vec{p} = q\vec{d}$

& if each dipole occupies a surface area S, the dipole moment per unit area is $\vec{\tau} = \vec{p}/S$

$$\vec{\tau} = \frac{\vec{p}}{S} = \frac{q\vec{d}}{S} = \left(\frac{q}{S}\right)\vec{d} = \underbrace{\sigma}_{\text{surface charge density}}\vec{d}$$



A "dipole layer" occurs if we take the limit $d \to 0$ & $\sigma \to \infty$ with $\tau = \sigma d$ held constant.

→ The potential from a dipole layer:

more generally we can express $\vec{\tau} = \frac{d\vec{p}}{dS}$ where we are allowing the moment to vary in magnitude & direction over the surface

the potential follows from a superposition of the potential from infinitesimal point dipoles

$$d\phi(\vec{r}) = -\frac{1}{4\pi\epsilon_0}\, d\vec{p}(\vec{r}_s)\cdot\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}_s|}$$

$$= -\frac{1}{4\pi\epsilon_0}\, dS\ \vec{\tau}(\vec{r}_s)\cdot\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}_s|}$$

$$\& \quad \phi(\vec{r}) = -\frac{1}{4\pi\epsilon_0}\int dS\ \vec{\tau}(\vec{r}_s)\cdot\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}_s|}$$

we can convert this to a volume integral using a delta function

$$\phi(\vec{r}) = -\frac{1}{4\pi\epsilon_0}\int d^3\vec{r}'\ \delta(z')\ \vec{\tau}(\vec{r}_s')\cdot\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|} = +\frac{1}{4\pi\epsilon_0}\int d^3\vec{r}'\ \delta(z')\,\vec{\tau}(\vec{r}_s')\cdot\vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}'|}$$

$$= -\frac{1}{4\pi\epsilon_0}\int d^3\vec{r}'\ \frac{\vec{\nabla}'\cdot\left(\delta(z')\vec{\tau}(\vec{r}_s')\right)}{|\vec{r}-\vec{r}'|}$$ after integrating by parts

$$\Rightarrow \rho(\vec{r}) = -\vec{\nabla}\cdot\left(\delta(z)\,\vec{\tau}(\vec{r}_s)\right)$$

if the dipoles are perpendicular to the surface, $\rho(\vec{r}) = -\tau_z(\vec{r}_s)\,\delta'(z)$

Poisson's eqn $-\nabla^2\phi = \rho/\varepsilon_0$ for slow variation with $\tilde{r}_s$ becomes $-\frac{d^2}{dz^2}\phi = \rho/\varepsilon_0$

or $\varepsilon_0 \frac{d^2\phi}{dz^2} = \tau_z \delta'(z)$

so integrating we get $\varepsilon_0 \frac{d\phi}{dz} = \tau_z \delta(z) + c$

& integrating again from $z=0^-$ to $z=0^+$: $\boxed{\phi(z=0^+) - \phi(z=0^-) = \tau_z/\varepsilon_0}$

& we see that the potential is discontinuous across a dipole layer

$\boxed{\text{SPHERICAL EXPANSION}}$

There is a very useful alternative way to expand $\dfrac{1}{|\vec{r}-\vec{r}\,'|}$ for $r > r'$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{\sqrt{r^2 - 2\vec{r}\cdot\vec{r}\,' + r'^2}} = \frac{1}{r}\left[1 - 2\cos\gamma\left(\frac{r'}{r}\right) + \left(\frac{r'}{r}\right)^2\right]^{-1/2} \qquad (\cos\gamma = \hat{r}\cdot\hat{r}')$$

The "generating function" for the Legendre Polynomials, $P_n(x)$ $\left[\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n(x)\right] + n(n+1)P_n(x) = \right.$
reads
$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} t^n P_n(x)$$

& thus it follows that

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\gamma) \qquad r' < r$$

orthogonality: $\displaystyle\int_{-1}^{1} dx\, P_n(x)\,P_m(x) = \frac{2}{2n+1}\delta_{nm}$

completeness: $\displaystyle\sum_{n=0}^{\infty} \frac{2n+1}{2} P_n(x)\,P_n(x') = \delta(x-x')$

$P_0(x) = 1$; $P_1(x) = x$; $P_2(x) = \frac{1}{2}(3x^2-1)$; ...

Another useful result comes if we express $\cos\gamma$ in terms of the angles (θ,ϕ), (θ',ϕ') of $\hat{r}$, $\hat{r}'$ relative to a fixed set of axes

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$

The "spherical harmonic addition theorem" states that

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi')\,Y_{lm}(\theta,\phi)$$

where the "spherical harmonics", $Y_{lm}(\theta,\phi)$, are the solutions of the differential equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial Y}{\partial\theta}\right] + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} + l(l+1)Y = 0.$$

They are orthonormal: $\displaystyle\int_0^{2\pi} d\phi \int_0^{\pi}\sin\theta\, d\theta\; Y_{lm}(\theta,\phi)\,Y^*_{l'm'}(\theta,\phi) = \delta_{ll'}\delta_{mm'}$

complete: $\displaystyle\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}(\theta,\phi)\,Y_{lm}(\theta',\phi') = \delta(\cos\theta-\cos\theta')\,\delta(\phi-\phi') = \frac{1}{\sin\theta}\delta(\theta'-\theta)\,\delta(\phi'-\phi)$

and have the following behavior under conjugation: $Y^*_{lm}(\theta,\phi) = (-1)^m Y_{l,-m}(\theta,\phi)$

$Y_{00}(\theta,\phi) = \dfrac{1}{\sqrt{4\pi}}$; $Y_{10}(\theta,\phi) = \sqrt{\dfrac{3}{4\pi}}\cos\theta = \sqrt{\dfrac{3}{4\pi}}\,z/r$

...

$Y_{1\pm1}(\theta,\phi) = \mp\sqrt{\dfrac{3}{8\pi}}\sin\theta\, e^{\pm i\phi} = \mp\sqrt{\dfrac{3}{8\pi}}\,\dfrac{x \pm iy}{r}$

it follows that we obtain the expansion $$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r}\sum_{\ell=0}^{\infty}\frac{4\pi}{2\ell+1}\left(\frac{r'}{r}\right)^{\ell}\sum_{m=-\ell}^{\ell} Y_{\ell m}^{*}(\theta'\phi')\, Y_{\ell m}(\theta,\phi)$$
(for $r'<r$)

the symmetry of $r \leftrightarrow r'$ is such that we can easily obtain the expansion valid in the case $r<r'$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r'}\sum_{\ell=0}^{\infty}\frac{4\pi}{2\ell+1}\left(\frac{r}{r'}\right)^{\ell}\sum_{m=-\ell}^{\ell} Y_{\ell m}^{*}(\theta,\phi)\, Y_{\ell m}(\theta'\phi') \quad (\text{for } r<r')$$

SPHERICAL MULTIPOLE EXPANSION

Let's again consider a distribution of charge entirely within a sphere of radius R
- the potential anywhere is given by

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3\vec{r}\,' \frac{\rho(\vec{r}\,')}{|\vec{r}\,'-\vec{r}|}$$

now considering locations $\vec{r}$ far from the charge distribution ($r>>R>r'$) we can use our expansion

$$\frac{1}{|\vec{r}\,'-\vec{r}|} = \frac{1}{r}\sum_{\ell}\frac{4\pi}{2\ell+1}\left(\frac{r'}{r}\right)^{\ell}\sum_{m} Y^{*}_{\ell m}(\theta',\phi')\,Y_{\ell m}(\theta,\phi)$$

to obtain the "exterior spherical expansion"

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{\ell m} A_{\ell m}\,\frac{Y_{\ell m}(\theta,\phi)}{r^{\ell+1}} \qquad \text{with } A_{\ell m} \equiv \frac{4\pi}{2\ell+1}\int d^3\vec{r}\,'\rho(\vec{r}\,')\,r'^{\ell}\,Y^{*}_{\ell m}(\theta',\phi')$$

being the "exterior spherical multipole moments"

note that these objects are related to the cartesian moments we saw previously,

e.g. $$A_{1,\pm1} = \frac{4\pi}{3}\int d^3\vec{r}\,\rho(\vec{r})\,r\left(\mp\sqrt{\frac{3}{8\pi}}\right)\frac{x\pm iy}{r}$$

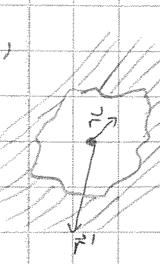
$$A_{1,0} = \frac{4\pi}{3}\int d^3\vec{r}\,\rho(\vec{r})\,r\sqrt{\frac{3}{4\pi}}\,z/r$$

$$\Rightarrow \quad A_{1,+1} - A_{1,-1} = -\sqrt{\frac{4\pi}{3}}\cdot\sqrt{2}\int d^3\vec{r}\,\rho(\vec{r})\,x = -\sqrt{\frac{8\pi}{3}}\,p_x$$

$$A_{1,+1} + A_{1,-1} = -i\sqrt{\frac{4\pi}{3}}\,\sqrt{2}\int d^3\vec{r}\,\rho(\vec{r})\,y = -i\sqrt{\frac{8\pi}{3}}\,p_y$$

$$A_{1,0} = \sqrt{\frac{4\pi}{3}}\int d^3\vec{r}\,\rho(\vec{r})\,z = \sqrt{\frac{4\pi}{3}}\,p_z$$

We can also define an expansion valid within a hole in a charge distribution, the "interior spherical expansion", where $r<r'$,



$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{\ell m} B_{\ell m}\,r^{\ell}\,Y^{*}_{\ell m}(\theta,\phi)$$

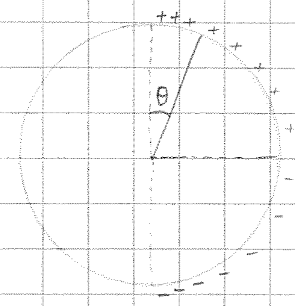
$$\text{with } B_{\ell m} \equiv \frac{4\pi}{2\ell+1}\int d^3\vec{r}\,'\frac{\rho(\vec{r}\,')}{r'^{\ell+1}}\,Y_{\ell m}(\theta',\phi')$$

an example of the use of the multipole expansion:

Consider a spherical shell of radius R, centered at the origin, carrying a surface charge density of

$$\sigma(\theta) = \sigma_0 \cos\theta .$$

Find the potential at all points inside & outside of the sphere.



outside the sphere $(r>R)$ we expand in an "exterior" spherical expansion with coefficients

$$A_{\ell m} = \frac{4\pi}{2\ell+1} \int d^3\vec{r}\,' \rho(\vec{r}\,') r'^{\ell} Y^*_{\ell m}(\theta',\phi')$$

since the charge is confined to the spherical surface of radius R

$$A_{\ell m} \longrightarrow \frac{4\pi}{2\ell+1} \int dS \; R^{\ell} \sigma(\theta) Y^*_{\ell m}(\theta,\phi)$$

& since $\sigma(\theta)$ is independent of ϕ

$$A_{\ell m} = \frac{4\pi}{2\ell+1} R^{\ell} \int R^2 \sin\theta \, d\theta \; \sigma(\theta) \int d\phi \, Y^*_{\ell m}(\theta,\phi)$$

$$Y_{\ell m}(\theta,\phi) \sim e^{im\phi}$$

$$\Rightarrow \int d\phi \, Y_{\ell m}(\theta,\phi) \propto \delta_{m0}$$

$$Y_{\ell 0}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta)$$

$$= \sqrt{\frac{4\pi}{2\ell+1}} R^{\ell+2} \int \sin\theta \, d\theta \; \sigma(\theta) P_\ell(\cos\theta) 2\pi \delta_{m0}$$

$$= \sqrt{\frac{4\pi}{2\ell+1}} R^{\ell+2} \sigma_0 \int d\theta \; \sin\theta \cos\theta \, P_\ell(\cos\theta) 2\pi \delta_{m0}$$

$$\cos\theta = P_1(\cos\theta)$$

$$= \sqrt{\frac{4\pi}{2\ell+1}} R^{\ell+2} \sigma_0 \int d(\cos\theta) \; P_1(\cos\theta) P_\ell(\cos\theta) 2\pi \delta_{m0}$$

$$= \sqrt{\frac{4\pi}{2\ell+1}} R^{\ell+2} \sigma_0 \cdot \frac{2}{2\ell+1} \delta_{\ell 1} \delta_{m0} \, 2\pi$$

$$\Rightarrow \text{all } A_{\ell m} = 0 \quad \text{except} \quad A_{10} = \frac{2\sqrt{4\pi}}{3\sqrt{3}} \sigma_0 R^3 \, 2\pi$$

$$\& \quad \phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell m} A_{\ell m} \frac{Y_{\ell m}(\theta,\phi)}{r^{\ell+1}} = \frac{2\pi}{4\pi\epsilon_0} \frac{2\sqrt{4\pi}}{3\sqrt{3}} \sigma_0 R^3 \cdot \frac{\sqrt{3}}{\sqrt{4\pi}} P_1(\cos\theta) \cdot \frac{1}{r^2}$$

$$\underline{\phi(\vec{r}) = \frac{\sigma_0}{3\epsilon_0} \frac{R^3}{r^2} \cos\theta \qquad \text{for all } r>R}$$

inside the sphere, $r<R$, we can perform an "interior multipole expansion" with coefficients

$$B_{\ell m} = \frac{4\pi}{2\ell+1} \int d^3\vec{r}\,' \frac{\rho(\vec{r}\,')}{r'^{\ell+1}} Y_{\ell m}(\theta'\phi')$$

$$\rightarrow \frac{4\pi}{2\ell+1} \frac{1}{R^{\ell+1}} R^2 \int d\phi \int d(\cos\theta)\ \sigma(\theta)\, Y_{\ell m}(\theta,\phi)$$

$$= \frac{4\pi}{2\ell+1} \frac{1}{R^{\ell-1}} 2\pi\delta_{m0} \int d(\cos\theta)\ \sigma_0\cos\theta \cdot \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)$$

$$B_{\ell m} = \frac{(4\pi)^{3/2}}{3\sqrt{3}} \sigma_0\, \delta_{\ell 1}\delta_{m0}$$

$$\Rightarrow \phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{\ell m} B_{\ell m} r^\ell Y^*_{\ell m}(\theta,\phi) = \underline{\frac{\sigma_0}{3\varepsilon_0} r\cos\theta} \quad \text{for } r<R$$

$$\phi(\vec{r}) = \frac{\sigma_0}{3\varepsilon_0} \begin{cases} r\cos\theta & r<R \\ \frac{R^3}{r^2}\cos\theta & r>R \end{cases}$$

inside the sphere $\phi = \frac{\sigma_0}{3\varepsilon_0} z \Rightarrow \vec{E} = -\vec{\nabla}\phi = -\frac{\sigma_0}{3\varepsilon_0}\hat{z}$

outside the sphere $\phi = \frac{\left(\frac{4\pi}{3}\sigma_0 R^3\right)}{4\pi\varepsilon_0} z/r^3 \Rightarrow$ sphere behaves as an electric dipole of moment $p = \frac{4\pi}{3}R^3\sigma_0$

* exercise: check that $\vec{p} \equiv \int d^3\vec{r}\,'\ \vec{r}\,' \rho(\vec{r}\,')$ in this case also gives a dipole moment of magnitude $\frac{4\pi}{3}R^3\sigma_0$ directed along $\hat{z}$