# PERFECT CONDUCTORS

A "perfect" conductor is defined by the property that everywhere inside its volume the electric field is zero and there is zero charge density.

Any excess charge placed on a perfect conductor will accumulate on the surface.*

e.g. a conducting sphere of radius $R$ has a charge $Q$ placed on it - the charge will spread itself out evenly to provide a uniform surface charge density of $\sigma = Q/4\pi R^2$

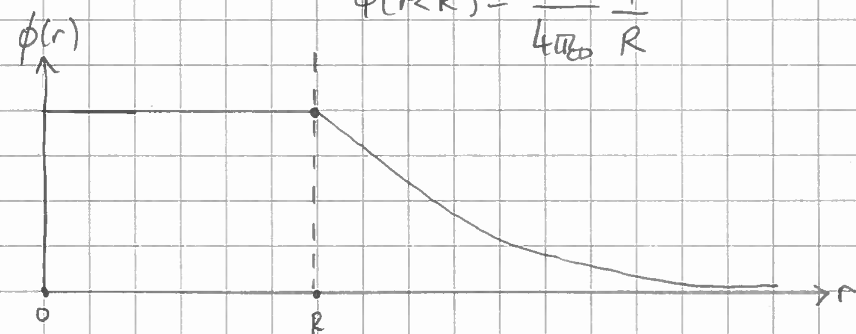
Gauss's law indicates that outside the sphere ($r>R$) the electric field is exactly the same as from a point charge $Q$ at the origin

$$\vec{E}(r>R) = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2} \quad \Rightarrow \quad \phi(r>R) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}$$

inside the sphere the electric field is zero

$\vec{E}(r<R) = 0$ and so the constant potential which matches at the boundary is

$$\phi(r<R) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{R}$$

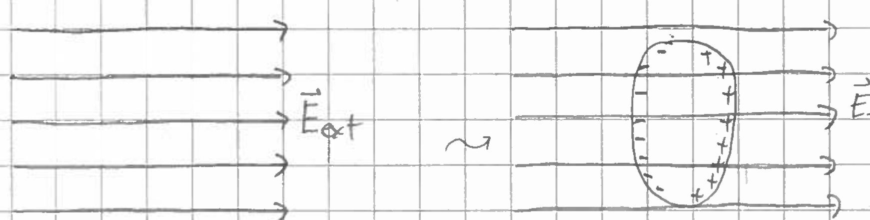


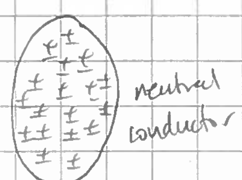

* recall the homework problem where the minimum energy for charge lying between two spheres was achieved if all the charge lies on the outermost surface

The basic idea behind perfect conductors is that the +/- charges naturally present in equal numbers in the conductor are "completely free to move" under the influence of any applied external electric field.

They will rapidly* rearrange themselves in such a way as to ensure that $\vec{E} = \vec{0}$ inside the conductor. The resulting distribution of charge is over the surface of the conductor

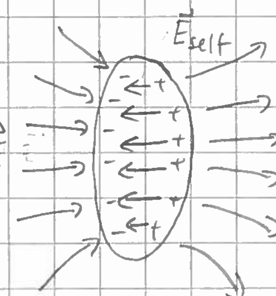
* infinitely fast in a *perfect* conductor
$\mathcal{O}(10^{-19}\,\mathrm{s})$ in copper

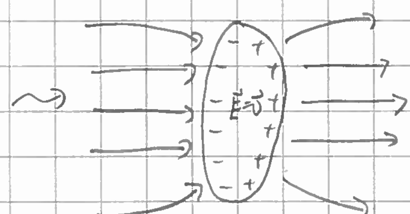



charges rearrange under force provided by $\vec{E}_{ext}$

$\to$ produces a "self" $\vec{E}$




total field is the sum of the external & self fields & is zero inside the conductor.

$$\vec{E}(\vec{r}) = \vec{E}_{ext} + \vec{E}_{self} = \vec{E}_{ext} + \frac{1}{4\pi\varepsilon_0}\int_S dS'\,\sigma(\vec{r}\,')\,\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

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recall the matching conditions for $\vec{E}$ at a surface charge

$\to$ field parallel to surface is continuous : $\hat{n}\times(\vec{E}_1 - \vec{E}_2) = 0$

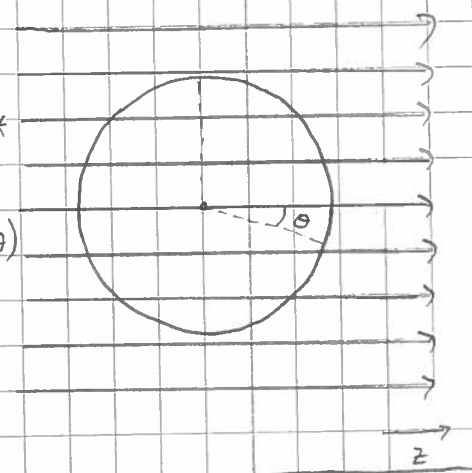
$\to$ field perpendicular to surface is discontinuous $\hat{n}\cdot(\vec{E}_2 - \vec{E}_1) = \sigma/\varepsilon_0$ ($\hat{n}$ points from 1 to 2)

thus at the surface of a conductor with $\vec{E}_1 = \vec{E}_{in} = \vec{0}$ & $\hat{n}$ point outward
$\vec{E}_2 = \vec{E}_{out}$

$\hat{n}\times\vec{E}_{out} = 0$ $\Rightarrow$ field just outside a conductor is normal to the surface

& $\vec{E}_{out} = \sigma/\varepsilon_0\,\hat{n}$

an illustrative example: a conducting sphere in a uniform applied electric field




→ sphere of radius $R$ at the origin

→ applied external field along the z-axis : $\vec{E}_{ext} = E_0 \hat{z}$ *

$\Rightarrow \phi_{ext} = -E_0 z = -E_0 r \cos\theta$

the external field will generate a surface charge distribution $\sigma(\theta)$

• inside the sphere the self-generated field must cancel $\vec{E}_{ext}$

$$\Rightarrow \vec{E}_{self} = -E_0 \hat{z} \quad \Rightarrow \phi_{self}(r<R) = +E_0 r \cos\theta$$

* choosing the zero of potential so that $\phi(0)=0$

• outside the sphere, $\sigma(\theta)$ will produce a field $\vec{E}_{self}$ that we can expand in an exterior spherical multipole expansion

$$\phi_{self}(r>R) = \frac{1}{4\pi\varepsilon_0} \sum_{lm} A_{lm} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

but since the system has azimuthal symmetry (independent of $\phi$)

$$\phi_{self}(r>R) = \frac{1}{4\pi\varepsilon_0} \sum_l A_l \frac{\sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta)}{r^{l+1}} = \frac{A}{r} + \frac{B}{r^2}\cos\theta + \frac{C}{r^3} P_2(\cos\theta) + \dots$$

the potential due to the surface charge distribution $\sigma(\theta)$ must be continuous at the surface of the sphere

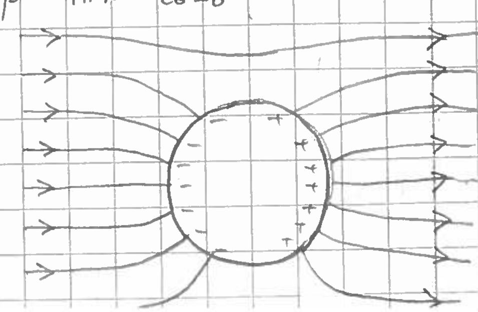
$$\phi_0 - E_0 R\cos\theta = \frac{A}{R} + \frac{B}{R^2}\cos\theta + \frac{C}{R^3} P_2(\cos\theta) + \dots$$

so that, using the orthogonality of the Legendre polynomials : $0 = A/R$, $\frac{B}{R^2} = +E_0 R$

$$\& \text{ thus } \phi_{self}(r>R) = \frac{E_0 R^3}{r^2}\cos\theta = \frac{1}{4\pi\varepsilon_0} \cdot (4\pi\varepsilon_0 E_0 R^3) \cdot \frac{z}{r^3}$$

$\Rightarrow$ the surface charge density produces an external dipole field with dipole moment $p = 4\pi R^3 \cdot \varepsilon_0 E_0$

$$\vec{E}(r>R) = E_0 \hat{z} + -\vec{\nabla}\phi_{self}$$



the applied electric field has "polarized" the sphere - we can define a constant, $\alpha$, which describes how easily a body is polarized

$$\vec{p} = \alpha \, \varepsilon_0 \vec{E}_0 \qquad \alpha = \text{"polarizibility"}$$

& in this case $\alpha = 4\pi R^3$ - in general $\alpha \propto$ (volume)

we can easily find the surface charge density induced using

$$\vec{E}_{out} = \frac{\sigma}{\varepsilon_0}\hat{n} \qquad \Rightarrow \sigma = \varepsilon_0 \hat{r} \cdot (-\vec{\nabla}\phi_{out}) = -\varepsilon_0 \frac{\partial}{\partial r}\phi_{out}\Big|_{r=R}$$

$$\phi_{out} = -E_0 r\cos\theta + E_0 \frac{R^3}{r^2}\cos\theta$$

$$\frac{\partial\phi_{out}}{\partial r} = -E_0\cos\theta - 2E_0\frac{R^3}{r^2}\cos\theta \qquad \Rightarrow \underline{\sigma = 3\varepsilon_0 E_0 \cos\theta}$$

## SCREENING & SHIELDING

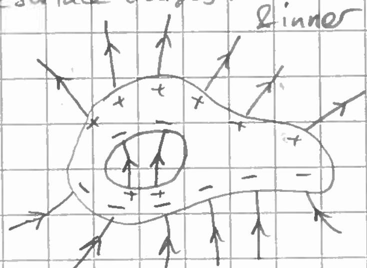
Consider a perfect conductor which contains a vacuum cavity.

We know that even when placed in an external electric field, the electric field at every point in the conductor is zero.

But it is also the case that the electric field in the cavity is <u>also</u> zero.

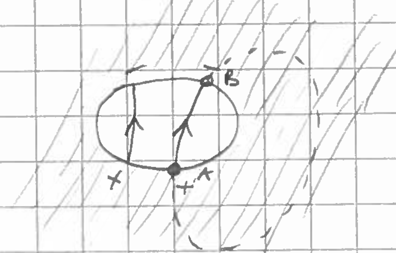
We can prove this by assuming that the field in the cavity is not zero & then encountering an inconsistency:

propose surface charges on outer & inner



perform a line integral as shown

– the part in the cavity goes along a field-line



$$0 = \oint d\vec{l} \cdot \vec{E} = \int_A^B dl\, |\vec{E}_{cav}| + \int_B^A d\vec{l} \cdot \underbrace{\vec{E}_{cond}}_{\to 0}$$

$$0 = \int_A^B dl\, |\vec{E}_{cav}|$$

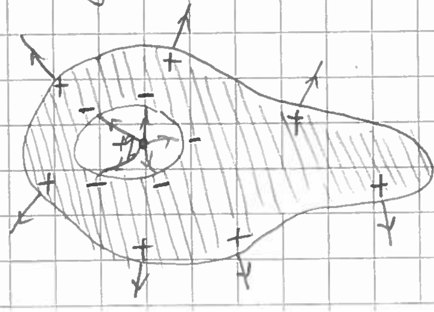
$$\Rightarrow \underline{\vec{E}_{cav} = \vec{0}}$$

this also ensures that there can be no surface charge distribution on the cavity wall, as that would generate an $\vec{E}_{cav}$.

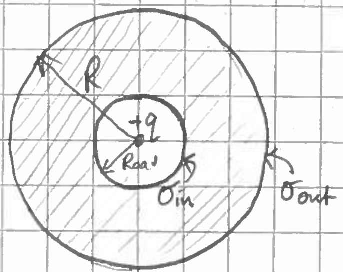
It follows that the cavity is completely <u>shielded</u> from the effects of any charges placed outside the conductor.

Surface charge can accumulate on the cavity wall in the case that a charge is placed <u>inside</u> the cavity

& an $\vec{E}$-field appears outside the conductor.



e.g. Spherical geometry




spherical gaussian surface inside the cavity

$$\int dS\, E_{cav} = q/\varepsilon_0 \qquad E_{cav} = \frac{q}{4\pi\varepsilon_0 r^2}$$

spherical gaussian surface inside the conductor

$$\underbrace{\int dS\, E_{cond}}_{0} = \frac{q + \int_{cav} dS\, \sigma_{in}}{\varepsilon_0}$$

$$\Rightarrow \sigma_{in} = -q/4\pi R_{cav}^2$$

Spherical gaussian surface outside the conductor

$$\int dS\, E_{out} = \frac{1}{\varepsilon_0}\Big(q + \underbrace{Q_{in}}_{=-q} + \int_{cond} dS\, \sigma_{out}\Big) = \frac{1}{\varepsilon_0} Q_{out}$$

but all the charge, $Q_{in}$, that appeared on the inner surface must be compensated by the charge on the outer surface, $Q_{out}$

$$\Rightarrow Q_{out} = -Q_{in} = -(-q) = q$$

$$\Rightarrow E_{out} = q/4\pi\varepsilon_0 r^2$$ exactly as though the conductor weren't there.

## CAPACITANCE & "GROUNDING"

We can define a geometric quantity, the "self-capacitance" which measures how good a conductor is at storing charge:

Suppose an isolated conductor has a surface $S$ containing a total charge $Q$. All points on $S$ are at the same potential $\phi(\vec{r}_s) = V$, as are all points inside $S$.

Outside the surface $S$, the potential will fall, so $\phi(\vec{r}) = V\tilde{\phi}(\vec{r})$ with $\tilde{\phi}(r\to\infty)\to 0$ & $\tilde{\phi}(\text{cond}) = 1$

then Gauss's law over the surface $S$ reads

$$Q/\varepsilon_0 = \int d\vec{S}\cdot\vec{E} = -\int d\vec{S}\cdot\vec{\nabla}\phi = -V\int d\vec{S}\cdot\vec{\nabla}\tilde{\phi}$$

& we define the capacitance $\underline{C = Q/V = -\varepsilon_0\int d\vec{S}\cdot\vec{\nabla}\tilde{\phi}}$

e.g. its easy to show that a spherical conducting shell, or a solid conducting sphere have a capacitance of

$$C = 4\pi\varepsilon_0 R$$

typically we expect a surface of area $A$ to have a capacitance of roughly $C \simeq \varepsilon_0\sqrt{4\pi A}$

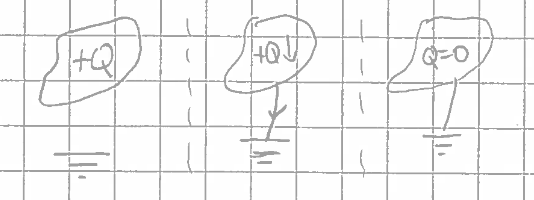
The Earth can be considered to be a very large, nearly spherical, conductor, whose self-capacitance is huge: $R_E = 6.4\times10^6\,\text{m}$, $\varepsilon_0 = 8.9\times10^{-12}\,\text{F/m}$ $\Rightarrow C_E \simeq 7.2\times10^{-4} \simeq 0.7\,\text{mF}$

This means that we can take charge from the Earth, or place charge on the Earth, without significantly changing its potential.

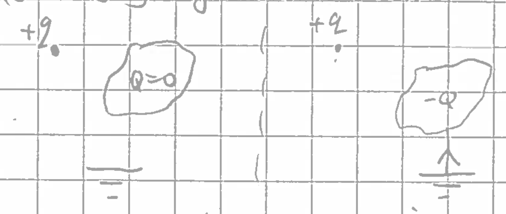
e.g. to change the potential of the Earth by $1\,\text{V}$ requires about $1\,\text{mC}$

We usually approximate by saying $C_E\to\infty$ & $\phi_E = 0$ no matter how much charge we exchange. The Earth then acts as a reservoir of charge.

(a) "grounding a charged conductor"




(b) "charging a grounded conductor"




charge +Q can "spread" out in the

Coulomb's attraction pulls negative charge out of ground

The force on a conductor is easy to find. Since all the charge lies on the surface, we can use the result we derived earlier,

$$\frac{d\vec{F}}{dS} = \sigma \cdot \frac{1}{2}(\vec{E}_{in} + \vec{E}_{out})$$

where since we're considering the outer surface of a conductor we must have $\vec{E}_{in} = \vec{0}$

$$\& \quad \vec{E}_{out} = \frac{\sigma}{\varepsilon_0}\hat{n} \qquad \text{so that} \quad \frac{d\vec{F}}{dS} = \frac{\sigma^2}{2\varepsilon_0}\hat{n}$$

and integrating over the surface : $\vec{F} = \int dS \, \frac{\sigma^2(\vec{r}_s)}{2\varepsilon_0}\hat{n} = \frac{1}{2\varepsilon_0}\int d\vec{S} \; \sigma^2(\vec{r}_s)$

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There's a work-energy argument which leads to the same result:


the energy density just outside a conductor is $W = \frac{1}{2}\varepsilon_0 |\vec{E}|^2 = \frac{1}{2}\varepsilon_0\left(\frac{\sigma}{\varepsilon_0}\right)^2 = \sigma^2/2\varepsilon_0$.

Now suppose we move an infinitesimal area of surface a distance $dx$ outwards, now there is no electric field in a volume $dS dx$ & thus the energy stored in the $\vec{E}$-field has decreased by $\frac{\sigma^2}{2\varepsilon_0} dS dx$

we can consider this to be work done by the force on charge $\sigma dS$ : $dF dx$

$$dF\,dx = \frac{\sigma^2}{2\varepsilon_0} dS\,dx \quad \Rightarrow \quad \underline{\frac{dF}{dS} = \sigma^2/2\varepsilon_0}$$