DIELECTRIC MATTER

In general, materials do not behave like perfect conductors; they do not completely screen charges from their interior, and in response to an external electric field, while charges do rearrange themselves, they do not lead to a net zero field inside the material.

We talk of the POLARIZATION of a dielectric material in response to an external electric field. A simple (but not generally applicable) model of polarization, due to Lorentz, considers a volume distribution of point electric dipoles:

consider a volume ΔV containing
- "free" charges $Q_f = \rho_f(\vec{r}')\,\Delta V$
- point electric dipoles : net dipole moment $\Delta\vec{p} = \vec{P}\,\Delta V$

where we've used the Lorentz definition of polarization $\vec{P} = \dfrac{\Delta\vec{p}}{\Delta V}$ = volume density of dipole moments

the potential from these charges is

$$\Delta\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\rho_f(\vec{r}')}{|\vec{r}-\vec{r}'|}\Delta V + \left(\frac{-1}{4\pi\varepsilon_0}\right)\Delta\vec{p}\cdot\vec{\nabla}_{\vec{r}}\frac{1}{|\vec{r}-\vec{r}'|}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\rho_f(\vec{r}')}{|\vec{r}-\vec{r}'|}\Delta V - \frac{1}{4\pi\varepsilon_0}\vec{P}\cdot\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|}\Delta V$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{\rho_f(\vec{r}')}{|\vec{r}-\vec{r}'|} + \vec{P}\cdot\vec{\nabla}_{\vec{r}'}\frac{1}{|\vec{r}-\vec{r}'|}\right]\Delta V \qquad \text{using } \vec{\nabla}_{\vec{r}}\frac{1}{|\vec{r}-\vec{r}'|} = -\vec{\nabla}_{\vec{r}'}\frac{1}{|\vec{r}-\vec{r}'|}$$

integrating over a distribution, we'd get a net potential

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3\vec{r}'\left[\frac{\rho_f(\vec{r}')}{|\vec{r}-\vec{r}'|} + \vec{P}\cdot\vec{\nabla}_{\vec{r}'}\frac{1}{|\vec{r}-\vec{r}'|}\right]$$

& integrating by parts in the second term

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3\vec{r}'\,\frac{\left(\rho_f(\vec{r}') - \vec{\nabla}\cdot\vec{P}(\vec{r}')\right)}{|\vec{r}-\vec{r}'|} + \frac{1}{4\pi\varepsilon_0}\underbrace{\int d^3\vec{r}'\,\vec{\nabla}_{\vec{r}'}\cdot\left[\vec{P}(\vec{r}')\frac{1}{|\vec{r}-\vec{r}'|}\right]}_{\text{div. theorem}}$$

$$\oint d\vec{S}'\cdot\vec{P}(\vec{r}')\frac{1}{|\vec{r}-\vec{r}'|}$$

$$\text{(A)}\qquad \phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3\vec{r}'\,\frac{\rho_f(\vec{r}')}{|\vec{r}-\vec{r}'|} + \underbrace{\frac{1}{4\pi\varepsilon_0}\int d^3\vec{r}'\,\frac{-\vec{\nabla}\cdot\vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|}}_{\text{"volume polarization charge"}} + \underbrace{\frac{1}{4\pi\varepsilon_0}\oint dS\,\frac{\hat{n}(\vec{r}_s)\cdot\vec{P}(\vec{r}_s)}{|\vec{r}-\vec{r}_s|}}_{\text{"surface polarization charge"}}$$

the potential due to the polarization is then

$$\phi_p(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3\vec{r}\,' \frac{\rho_p(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} + \frac{1}{4\pi\epsilon_0}\int dS \frac{\sigma_p(\vec{r}_s)}{|\vec{r}-\vec{r}_s|} \quad \text{with} \quad \begin{aligned}\rho_p(\vec{r}) &= -\operatorname{div}\vec{P}(\vec{r})\\ \sigma_p(\vec{r}_s) &= \hat{n}(\vec{r}_s)\cdot\vec{P}(\vec{r}_s)\end{aligned}$$

& this result is actually more general than the Lorentz model derivation suggests (see Zangwill for more discussion)

A simple example: consider a sphere of radius R, having uniform polarization $\vec{P}$.

Since the polarization is uniform, $\operatorname{div}\vec{P} = 0 \Rightarrow \rho_p(\vec{r}) = 0$,
but the surface charge is not zero: $\sigma_p(\vec{r}_s) = \hat{r}\cdot\vec{P}$

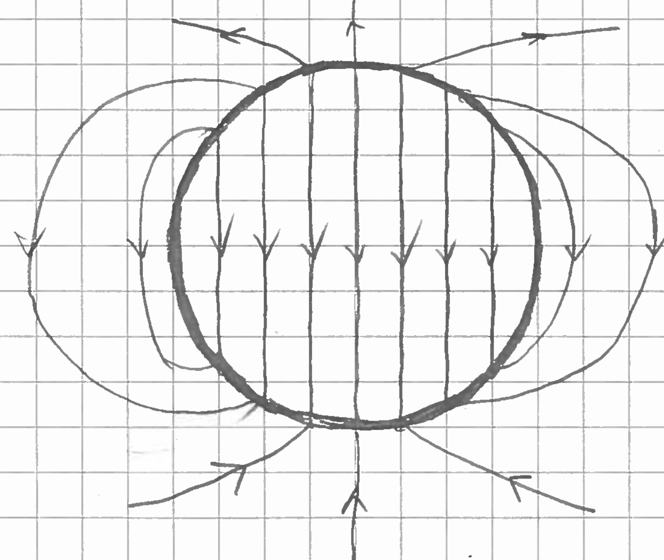
choosing the z-axis to be along $\vec{P}$: $\sigma_p = P\cos\theta$

but we already found the potential from such a surface charge distribution:

$$\phi(r,\theta) = \frac{P}{3\epsilon_0}\begin{cases} r\cos\theta & r<R \\ R^3/r^2\cos\theta & r>R\end{cases}$$

$$\vec{E} = -\vec{\nabla}\phi = \begin{cases} -\frac{1}{3\epsilon_0}\vec{P} & r<R \\ \frac{\left(\frac{4}{3}\pi R^3\right)}{4\pi\epsilon_0}\left[\frac{3(\hat{r}\cdot\vec{P})\hat{r}-\vec{P}}{r^3}\right] & r>R\end{cases}$$

$\leftarrow$ the field from an electric dipole moment $\vec{p} = \left(\frac{4}{3}\pi R^3\right)\vec{P}$



when both "free" charges & dielectric media are present, it is convenient to define an auxillary field, $\vec{D}$, sometimes called the "electric displacement"

$\vec{D} \equiv \varepsilon_0 \vec{E} + \vec{P}$ which is proportional to $\vec{E}$ outside dielectrics, but not within

from eqn (A) we have $\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}(\rho_f + \rho_P)$ with $\rho_P = -\mathrm{div}\,\vec{P}$

& thus $\vec{\nabla}\cdot\vec{D} = \rho_f - \vec{\nabla}\cdot\vec{P} + \vec{\nabla}\cdot\vec{P} = \rho_f$

$$\boxed{\vec{\nabla}\cdot\vec{D} = \rho_f}$$

& $\int_S d\vec{S}\cdot\vec{D} = Q_f$

it's still the case that $\vec{\nabla}\times\vec{E} = 0 \Rightarrow \vec{\nabla}\times\vec{D} = \vec{\nabla}\times\vec{P}$

then the Helmholtz theorem tells us that

$$\vec{D}(\vec{r}) = -\vec{\nabla}\int d^3\vec{r}\,'\,\frac{\rho_f(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|} + \vec{\nabla}\times\int d^3\vec{r}\,'\,\frac{\vec{\nabla}_{\vec{r}\,'}\times\vec{P}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}$$

& we see that non-zero $\vec{D}$ comes from:
- free charges
- curl-like spatial variations of $\vec{P}$.

Matching conditions can be derived using the same methods previously considered:

$\hat{n}\cdot(\vec{D}_2 - \vec{D}_1) = \sigma_f$ & $\hat{n}\times(\vec{E}_1 - \vec{E}_2) = 0$ ($\hat{n}$ points from 1 to 2)

normal component of $\vec{D}$ discontinuous by σ_{free}

transverse component of $\vec{E}$ is continuous the relationship between $\vec{P}$ & $\vec{E}$ can in general be quite complicated, e.g. experimental studies of dielectrics suggest that

$$P_i = \varepsilon_0 \chi_{ij} E_j + \varepsilon_0 \chi^{(2)}_{ijk} E_j E_k + \ldots$$

so $\vec{P}$ is not necessarily parallel to $\vec{E}$, nor is it always linear in $\vec{E}$

A particularly simple and common case is a LINEAR, ISOTROPIC dielectric where $\vec{P} = \varepsilon_0 \chi \vec{E}$ with the constant χ being known as the DIELECTRIC SUSCEPTIBILITY.

In this case we have $\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi \vec{E} = \varepsilon_0 (\chi + 1) \vec{E}$

& we can define the PERMITTIVITY, $\varepsilon \equiv \varepsilon_0 (1+\chi)$ & $\underline{\vec{D} = \varepsilon \vec{E}}$

the DIELECTRIC CONSTANT is the dimensionless ratio, $\underline{K = \varepsilon/\varepsilon_0} = 1 + \chi$

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e.g. polarizing a simple dielectric sphere & screening a dipole:

consider a point dipole of moment $\vec{p}_0$ embedded at the center of a linear, isotropic dielectric sphere of volume V and dielectric constant K. Find the total dipole moment of the system.

the dipole moment of the dielectric is $\vec{p}_P = \int_V d^3\vec{r}\, \vec{P} = \int_V d^3\vec{r}\, \varepsilon_0 (K-1) \vec{E} = \varepsilon_0 (K-1) \int_V d^3\vec{r}\, \vec{E}$

we proved earlier that in general the integral of $\vec{E}$ over a sphere is $\int_V d^3\vec{r}\, \vec{E} = -\vec{p}/3\varepsilon_0$

and thus $\vec{p}_P = \varepsilon_0 (K-1) \left[-\vec{p}/3\varepsilon_0 \right] \Rightarrow \vec{p} = \vec{p}_0 + \vec{p}_P = \vec{p}_0 - \vec{p} \cdot \frac{1}{3}(K-1)$

$$\vec{p}\left(1 + \tfrac{1}{3}(K-1)\right) = \vec{p}_0$$

$$\boxed{\vec{p} = \frac{3}{2+K} \vec{p}_0}$$

so for $K \geq 1$, the dielectric partially screens the embedded dipole moment

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in a LINEAR, ISOTROPIC dielectric: $\vec{\nabla} \cdot \vec{D} = \rho_f \Rightarrow \varepsilon_0 \vec{\nabla} \cdot \vec{E} + \vec{\nabla} \cdot \vec{P} = \varepsilon_0 \vec{\nabla} \cdot \vec{E} + (\varepsilon - \varepsilon_0) \vec{\nabla} \cdot \vec{E} = \varepsilon \vec{\nabla} \cdot \vec{E}$

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{E} = \rho_f/\varepsilon} \quad \& \quad \boxed{\nabla^2 \phi = -\rho_f/\varepsilon}$$

the polarization charge density, $\rho_P = -\vec{\nabla} \cdot \vec{P} = \left(\frac{1}{K} - 1\right) \rho_f$ $\Rightarrow$ no polarization charge density unless there's free charge

and the surface polarization charge, $\sigma_P = \vec{P} \cdot \hat{n}\big|_S = \varepsilon_0 (K-1) \vec{E} \cdot \hat{n}\big|_S$

DIELECTRIC RESPONSE TO CHARGE

→ Consider a point charge embedded in a simple dielectric medium.
the free charge density
$\rho_f(\vec{r}) = q\,\delta(\vec{r})$ induces a polarization charge density $\rho_p(\vec{r}) = \left(\frac{1}{K} - 1\right) q\,\delta(\vec{r})$

such that the total charge distribution is $\rho(\vec{r}) = \rho_f(\vec{r}) + \rho_p(\vec{r}) = \frac{q}{K}\delta(\vec{r})$

& the dielectric has screened the charge q down to a charge q/K.

Solving $\vec{\nabla}\cdot\vec{E} = \rho_f/\varepsilon$ the field from the charge is $\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon}\frac{\hat{r}}{r^2} = \frac{1}{K}\frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2} = \frac{1}{K}\vec{E}_{vac}(\vec{r})$

and the field is reduced by a factor $1/K$ w.r.t the value it would have in vacuo.

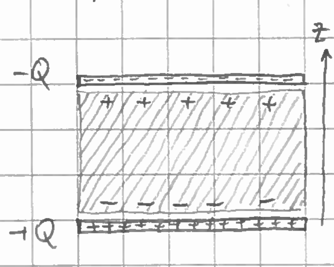
→ Consider a parallel plate capacitor with the plates, of area A, seperated by a distance, d carrying a charge Q. What happens if a dielectric is inserted to fill the space between the plates? [the free charge is held constant]

* without the dielectric there is an $\vec{E}$-field of $E_0 = \sigma_f/\varepsilon_0 = Q/\varepsilon_0 A$ between the plates

* with a dielectric we have $\vec{\nabla}\cdot\vec{D} = \rho_f \Rightarrow D = \sigma_f = Q/A$, but $D = \varepsilon E$ so $E = Q/\varepsilon A = E_0/K$

  and again the electric field strength has been reduced by $1/K$.

  this is due to the polarisation charge induced on the surface of the dielectric,
  $\sigma_p = \vec{P}\cdot\hat{n}$.



e.g. on the lower surface $\sigma_p = \vec{P}\cdot(-\hat{z}) = -P_z = -\varepsilon_0(K-1)E_z = -\varepsilon_0(K-1)\frac{\sigma_f}{\varepsilon}$

$$\sigma_p = \frac{1-K}{K}\sigma_f = \left(\frac{1}{K} - 1\right)\sigma_f$$

so the total surface charge on the bottom is $\sigma = \sigma_f + \sigma_p = \frac{1}{K}\sigma_f$

consider a point charge of embedded at the center of a dielectric sphere of rad , R.
Outside the sphere s vacuum.

Using $\vec{\nabla}\vec{D}$ ρ_f we have $\vec{D}(\vec{r})$ $\frac{q}{4\pi}\frac{\hat{r}}{r^2}$ everywhere.

ns de the sphere, $\vec{D} = \varepsilon\vec{E} \Rightarrow \vec{E}(<\ ) = \frac{q}{\tau}$ — $-E_{vac}$

de the sphere $\vec{D} = \varepsilon_0\vec{E} \Rightarrow \vec{E}(r>\ )$ $\frac{q}{\pi\varepsilon_0}\hat{r} = \vec{E}_{vac}$ ⇒ NO SCREENING!

Why doesn't the dielectric screen the charge outside the sphere?

Consider the surface charge induced on the outer surface of the sphere by polarization

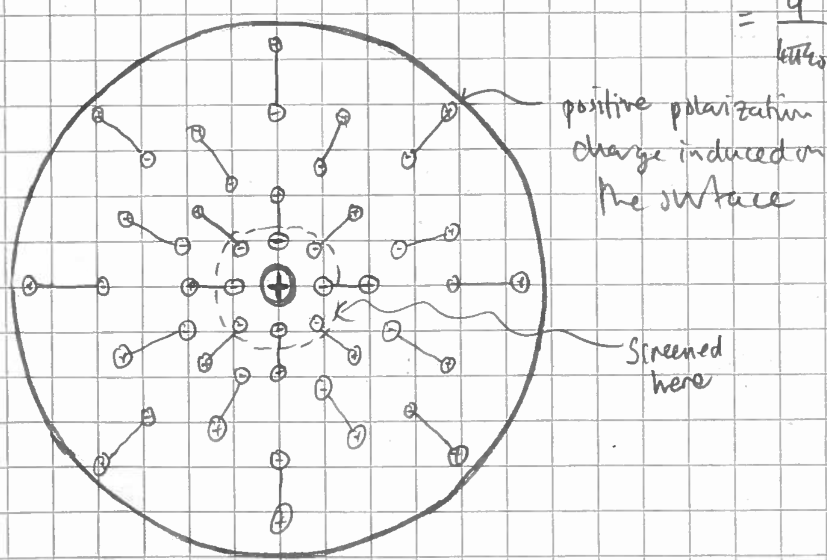
$$\sigma = \vec{P}\cdot\hat{n} = \hat{r}\cdot\vec{P} = \hat{r}\cdot\left(\varepsilon_0(\kappa-1)\vec{E}\right) = \varepsilon_0(\kappa-\ )\frac{}{4\pi\ \kappa\ ^2} = \left(1-\frac{1}{\kappa}\right)\frac{q}{r^2}$$

hile the charge does get screened locally $\vec{E}_q(r>R) = \frac{q}{4\pi\varepsilon}\frac{\hat{r}}{r^2}$

the charge induced on the dielectric boundary contributes $\vec{E}_p(r>R) = \left(1-\frac{1}{\kappa}\right)\frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$

so that $\vec{E}\ \ \vec{E}_q\ \vec{E}_p = \frac{1}{\kappa}\frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2} + \left(1-\frac{1}{\kappa}\right)\frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$

$$= \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$$



ENERGY IN DIELECTRICS

Let's try to find an expression for the energy stored in the fields in a polarized dielectric material.

Consider an infinite dielectric medium containing a charged conductor. Initially the conductor's surface carries a charge Q and is at potential ϕ.

the mechanical work required to transport an additional charge δQ from infinity to the conductor is $\phi . \delta Q$ & this energy gets stored in the $\vec{E}$-field set up in the dielectric

$$\delta U = \phi . \delta Q \qquad \text{but} \qquad \delta Q = \int_S d\vec{S} . \delta\vec{D} \qquad (\text{from } \vec{\nabla} . \vec{D} = \rho_f)$$

where $\delta\vec{D}$ is the extra $\vec{D}$ due to δQ.

& ϕ is constant over the surface of the conductor

$$\Rightarrow \delta U = \int_S d\vec{S} . \delta\vec{D}\, \phi$$

and the divergence theorem gives $\delta U = \int_{V_{cond}} d^3\vec{r}\ \vec{\nabla} . (\phi\, \delta\vec{D}) = \left(\int_{\substack{all\\space}} d^3\vec{r} - \int_{dielect} d^3\vec{r} \right) \vec{\nabla} . (\phi\, \delta\vec{D})$

& the integral over all space can be transformed into a surface integral at infinity which is zero

$$\delta U = - \int d^3\vec{r}\ \vec{\nabla} . (\delta\vec{D}\, \phi)$$

where the integral is over the dielectric volume

if there is no free charge in the dielectric then $\vec{\nabla} . \delta\vec{D} = 0$

$$\delta U = - \int d^3\vec{r}\ \delta\vec{D} . \vec{\nabla}\phi = \int d^3\vec{r}\ \vec{E} . \delta\vec{D}$$

& the total energy in the dielectric is the net work needed to establish $\vec{D}$:

$$U = \int d^3\vec{r} \int_0^{\vec{D}} \vec{E} . \delta\vec{D}$$

if the dielectric linear & isotropic, $\vec{D} = \varepsilon \vec{E}$ & $U = \int d^3\vec{r} \int_0^{\vec{D}} \frac{1}{\varepsilon} \vec{D} . \delta\vec{D} = \frac{1}{2} \int d^3\vec{r}\ \frac{1}{\varepsilon} |\vec{D}|^2$

or $\underline{U = \frac{1}{2} \int d^3\vec{r}\ \vec{E} . \vec{D}}$

notice that if the dielectric is actually vacuum we get back

$$U = \int d^3\vec{r}\ \frac{1}{2} \varepsilon_0 |\vec{E}|^2$$

another important energy is the change that occurs when a simple dielectric is inserted into an existing electric field, holding the charges producing $\vec{E}_0$ constant.

$$\text{egy before dielectric} = \frac{1}{2}\epsilon_0 \int d^3\vec{r}\, |\vec{E}_0|^2 = \frac{1}{2}\int d^3\vec{r}\, \vec{E}_0\cdot\vec{D}_0$$

$$\text{energy with dielectric} = \frac{1}{2}\int d^3\vec{r}\, \vec{E}\cdot\vec{D}$$

$$\Delta U = \frac{1}{2}\int d^3\vec{r}\,(\vec{E}\cdot\vec{D} - \vec{E}_0\cdot\vec{D}_0) = \frac{1}{2}\int d^3\vec{r}\left[(\vec{E}+\vec{E}_0)\cdot(\vec{D}-\vec{D}_0) + \vec{E}\cdot\vec{D}_0 - \vec{D}\cdot\vec{E}_0\right]$$

since $\vec{\nabla}\times\vec{E}=0$ & $\vec{\nabla}\times\vec{E}_0=0$, $\vec{\nabla}\times(\vec{E}+\vec{E}_0)=0$ & $\vec{E}+\vec{E}_0$ can be expressed as the gradient of a function,

$$\vec{E}+\vec{E}_0 = \vec{\nabla}\psi$$

$$\text{thus} \int d^3\vec{r}\,(\vec{E}+\vec{E}_0)\cdot(\vec{D}-\vec{D}_0) = \int d^3\vec{r}\,(\vec{\nabla}\psi)\cdot(\vec{D}-\vec{D}_0)$$

$$= \underbrace{\int d^3\vec{r}\,\vec{\nabla}\cdot\left[\psi(\vec{D}-\vec{D}_0)\right]}_{=0 \text{ (transfer to surface integral at finity)}} - \int d^3\vec{r}\,\psi\,\vec{\nabla}\cdot(\vec{D}-\vec{D}_0)$$

$$= -\int d^3\vec{r}\,\psi\left(\vec{\nabla}\cdot\vec{D} - \vec{\nabla}\cdot\vec{D}_0\right) = -\int d^3\vec{r}\,\psi(\vec{r})\underbrace{\left(\rho_f(\vec{r}) - \rho_f(\vec{r})\right)}_{=0}$$

$= 0$ since we kept the free charge constant.

$$\Delta U = \frac{1}{2}\int d^3\vec{r}\left[\vec{E}\cdot\vec{D}_0 - \vec{D}\cdot\vec{E}_0\right]$$

& since $\vec{D} = \epsilon_0\vec{E} + \vec{P}$

$$\boxed{\Delta U = -\frac{1}{2}\int d^3\vec{r}\,\vec{P}\cdot\vec{E}_0}$$