MAGNETIC INDUCTION & FARADAY'S LAW

Experimentally it is found that for a closed loop, C

$$\oint_C \vec{E}\cdot d\vec{\ell} = -\frac{d\Phi}{dt}$$

where Φ is the magnetic flux through the loop

$$\Phi = \int_S d\vec{S}\cdot\vec{B}$$

^any surface bounded by C

using Stokes' theorem we can write the LHS as $\int_S d\vec{S}\cdot\vec{\nabla}\times\vec{E}$

$$\text{so}\quad \int_S d\vec{S}\cdot\vec{\nabla}\times\vec{E} = -\frac{d}{dt}\int_S d\vec{S}\cdot\vec{B}$$

<u>IF</u> the path C is fixed in space (ie the loop is not moving w.r.t the magnetic field)

$$\text{then}\quad \int d\vec{S}\cdot\vec{\nabla}\times\vec{E} = -\int d\vec{S}\cdot\frac{\partial\vec{B}}{\partial t} \quad \& \quad \boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$$

"induction due to changing magnetic field"

if the path is moving we need to be more careful, but we will come to the same equation eventually

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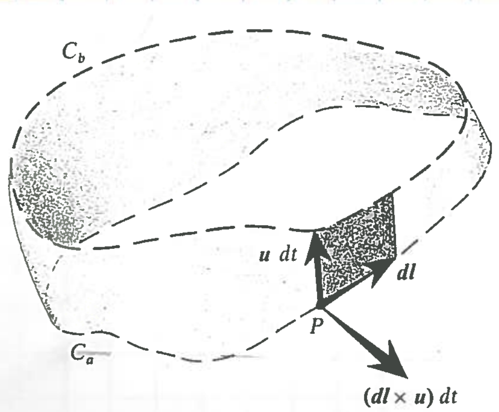

Figure 8-5. A path of integration moves from C_a to C_b in the time dt. The displacement is general and involves a translation, a rotation, and a distortion. The point P is assumed to move with a velocity $\boldsymbol{u}$ in a region where the magnetic induction is $\boldsymbol{B}$.

rate of change of flux

$$\frac{d\Phi}{dt} = \frac{\int_{S_b} d\vec{S}_b\cdot\vec{B}_b(t+dt) - \int_{S_a} d\vec{S}_a\cdot\vec{B}}{dt}$$

ie position the $\vec{B}$-field is measured at <u>AND</u> the time of measurement is changing consider the magnetic flux leaving the volume swept out during the time $t \to t+dt$, but evaluated at time $t+dt$:

$$\int \vec{B}\cdot d\vec{S} = \int_{S_b} d\vec{S}_b \cdot \vec{B}_b(t+dt) \overset{\text{outward going}}{-} \int_{S_a} d\vec{S}_a \cdot \vec{B}_a(t+dt) + \overset{\text{"side wall"}}{\int_{C_a} (d\vec{l}\times\vec{u}dt)\cdot\vec{B}(t+dt)}$$

divergence theorem

$$= \int d^3 r\, \vec{\nabla}\cdot\vec{B} = 0 \quad \text{since } \vec{\nabla}\cdot\vec{B} = 0$$

on surface S_a: $\vec{B}_a(t+dt)\cdot d\vec{S}_a = \vec{B}_a(t)\cdot d\vec{S}_a + dt\,\frac{\partial \vec{B}_a}{\partial t}\cdot d\vec{S}_a$ since S_a doesn't change with time

and $$\int_{C_a} (d\vec{l}\times\vec{u}dt)\cdot\vec{B}(t+dt) = dt \int_{C_a} (\vec{u}\times\vec{B}(t+dt))\cdot d\vec{l}$$

$$= dt \int d\vec{S}_a \cdot \vec{\nabla}\times(\vec{u}\times\vec{B}(t+dt)) \quad \text{using Stokes' theorem}$$

thus $$0 = \int_{S_b} d\vec{S}_b\cdot\vec{B}_b(t+dt) - \int_{S_a} d\vec{S}_a\cdot\vec{B}_a(t) - dt\int_{S_a} d\vec{S}_a\cdot\frac{\partial\vec{B}_a}{\partial t} + dt\int d\vec{S}_a\cdot\vec{\nabla}\times(\vec{u}\times\vec{B}(t)) + O(dt^2)$$

$$\& \frac{d\Phi}{dt} = \frac{dt\int d\vec{S}_a\cdot\left(+\frac{\partial\vec{B}_a}{\partial t} - \vec{\nabla}\times(\vec{u}\times\vec{B})\right)}{dt}$$

$$= -\int d\vec{S}\cdot\vec{\nabla}\times(\vec{u}\times\vec{B}) + \int d\vec{S}\cdot\frac{\partial\vec{B}}{\partial t}$$

$$\oint \vec{E}\cdot d\vec{l} = \int d\vec{S}\cdot\left(\vec{\nabla}\times(\vec{u}\times\vec{B}) + \frac{\partial\vec{B}}{\partial t}\right) \xrightarrow[\text{Stokes'}]{} \boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} + \vec{\nabla}\times(\vec{u}\times\vec{B})} \quad (u)$$

which looks like a different eqn! But we should be careful since this $\vec{E}$ is the electric field measured in the frame moving with velocity $\vec{u}$, which is <u>not</u> the same frame we're measuring $\vec{B}$ in.

in the "lab" frame, where the loop & all the charges in it are moving at velocity $\vec{u}$, the force on a charge q is

$$\vec{F}_0 = q\left(\vec{E}_0 + \vec{u}\times\vec{B}_0\right)$$

in the frame that moves along with the loop, the charges are at rest & feel a force

$$\vec{F}_u = q\,\vec{E}_u$$

assuming Gallilean invariance so that the force is the same in both frames we have

$$\vec{E}_u = \vec{E}_0 + \vec{u}\times\vec{B}$$

then eqn u $\longrightarrow$ $\vec{\nabla}\times\vec{E}_u = \vec{\nabla}\times\vec{E}_0 + \vec{\nabla}\times(\vec{u}\times\vec{B}) = -\dfrac{\partial\vec{B}}{\partial t} + \vec{\nabla}\times(\vec{u}\times\vec{B})$

& $\boxed{\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}}$ when $\vec{E}$ & $\vec{B}$ are measured in the same frame.

ENERGY IN THE MAGNETIC FIELD

In order to compute the energy stored in a system of steady-state currents and their associated magnetic fields, we need to consider the transient period during which we increase the currents and magnetic field from zero to their steady-state values. During this time, $\frac{\partial \vec{B}}{\partial t} \neq 0$ & there will be induced electromotive forces that cause the sources of current to do work.

The energy stored in the field is equal to the work done to set up the field.

Consider a closed loop of filamentary wire - in a time δt, the work done on a charge q_i by the induced electric field is $q_i \vec{E} \cdot \delta \vec{r}_i$ where $\delta \vec{r}_i = \vec{v}_i \delta t$ is the displacement of the charge. It follows that the work done by the source of the current is the negative of this

$$\delta W_{ext} = -\sum_i q_i \vec{E} \cdot \vec{v}_i \, \delta t \qquad \text{if we sum over the charge carriers}$$

Since $\vec{j}(\vec{r}) = \sum_i q_i \vec{v}_i \, \delta(\vec{r} - \vec{r}_i)$ for point charges we have

$$\delta W_{ext} = -\int d^3\vec{r} \; \vec{j}(\vec{r}) \cdot \vec{E}(\vec{r}) \, \delta t = -I \oint_C d\vec{\ell} \cdot \vec{E} \, \delta t \qquad \text{for a uniform filamentary wire}$$

If the wire is at rest $\oint_C d\vec{\ell} \cdot \vec{E} = -\frac{d}{dt} \int_S d\vec{S} \cdot \vec{B} = -\frac{d\Phi}{dt}$

& $\underline{\delta W_{ext} = I \, \delta \Phi}$ and this is the energy change $\underline{\delta U = I \, \delta \Phi}$

$$\delta U = I \, \delta \Phi = I \int_S d\vec{S} \cdot \delta \vec{B} = I \int_S d\vec{S} \cdot \vec{\nabla} \times \delta \vec{A} = I \oint_C d\vec{\ell} \cdot \delta \vec{A}$$

or since $I \oint d\vec{\ell} \longrightarrow \int d^3\vec{r} \; \vec{j}$ in general

$$\boxed{\delta U = \int d^3\vec{r} \; \vec{j} \cdot \delta \vec{A}}$$

Ampere's law $\vec{\nabla}\times\vec{H} = \vec{J}$

$$\delta U = \int d^3\vec{r}\ \delta\vec{A}\cdot(\vec{\nabla}\times\vec{H})$$

but $\vec{\nabla}\cdot(\vec{H}\times\delta\vec{A}) = \delta\vec{A}\cdot\vec{\nabla}\times\vec{H} - \vec{H}\cdot\vec{\nabla}\times\delta\vec{A}$

$\Rightarrow \delta\vec{A}\cdot(\vec{\nabla}\times\vec{H}) = \vec{\nabla}\cdot(\vec{H}\times\delta\vec{A}) + \vec{H}\cdot\delta\vec{B}$

$$= \underbrace{\int d\vec{S}\cdot\vec{H}\times\delta\vec{A}}_{\text{vanish far away}} + \int d^3\vec{r}\ \vec{H}\cdot\delta\vec{B}$$

$$\delta U = \int d^3\vec{r}\ \vec{H}\cdot\delta\vec{B}$$

(c.f $\int d^3\vec{r}\ \vec{E}\cdot\delta\vec{D}$ on page 79)

$\longrightarrow$ vacuum case $\vec{H} = \frac{1}{\mu_0}\vec{B} \Rightarrow U_{vac} = \frac{1}{2\mu_0}\int d^3\vec{r}\ |\vec{B}|^2$

$\longrightarrow$ linear medium $\vec{H} = \frac{1}{\mu}\vec{B} \Rightarrow U = \frac{1}{2}\int d^3\vec{r}\ \vec{H}\cdot\vec{B}$