MAGNETIZABLE MATERIAL

We'll treat matter here as a uniform collection of magnetic dipoles, and define MAGNETIZATION as the dipole moment per unit volume, $\vec{M} = \frac{\Delta\vec{m}}{\Delta V}$,

So the magnetic moment of an infinitesimal volume ΔV is $\Delta\vec{m} = \vec{M}\,\Delta V$, and if there is also a "free" current density of $\vec{J}$ in the same volume, the total vector potential due to this volume at a distant point is

volume ΔV at position $\vec{r}\,'$

$$\Delta\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{J}(\vec{r}\,')\Delta V}{|\vec{r}-\vec{r}\,'|} + \frac{\mu_0}{4\pi}\frac{\left(\vec{M}(\vec{r}\,')\Delta V\right)\times\left(\vec{r}-\vec{r}\,'\right)}{|\vec{r}-\vec{r}\,'|^3}$$

$$= \frac{\mu_0}{4\pi}\frac{\vec{J}(\vec{r}\,')\Delta V}{|\vec{r}-\vec{r}\,'|} + \frac{\mu_0}{4\pi}\Delta V\ \vec{M}(\vec{r}\,')\times\vec{\nabla}_{r'}\frac{1}{|\vec{r}-\vec{r}\,'|}$$

and adding up all contributions

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3\vec{r}\,'\left[\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} + \vec{M}(\vec{r}\,')\times\vec{\nabla}_{r'}\frac{1}{|\vec{r}-\vec{r}\,'|}\right]$$

$$\vec{\nabla}_{r'}\times\left[\frac{\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right] = \frac{1}{|\vec{r}-\vec{r}\,'|}\vec{\nabla}_{r'}\times\vec{M} - \vec{M}\times\vec{\nabla}_{r'}\frac{1}{|\vec{r}-\vec{r}\,'|}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3\vec{r}\,'\frac{\left(\vec{J}(\vec{r}\,') + \vec{\nabla}_{r'}\times\vec{M}(\vec{r}\,')\right)}{|\vec{r}-\vec{r}\,'|} - \frac{\mu_0}{4\pi}\int d\vec{S}\,'\times\frac{\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \quad (A)$$

we can define a magnetization current density $\vec{J}_M \equiv \vec{\nabla}\times\vec{M}$

and a magnetization surface current density $\vec{K}_M \equiv \vec{M}\times\hat{n}$

(where $\hat{n}$ is the "outward" normal to the surface

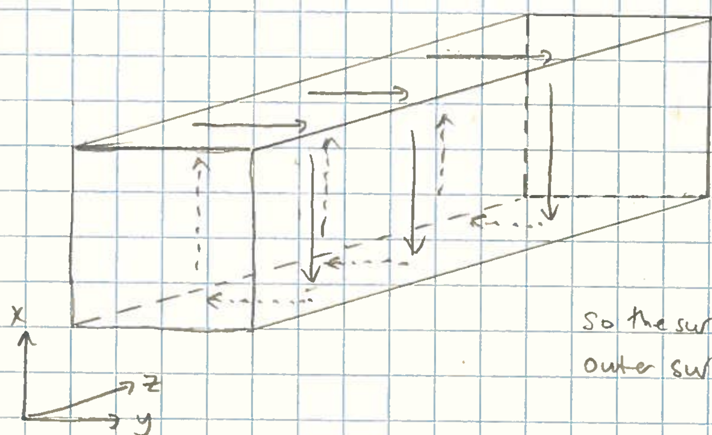
$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3\vec{r}\,'\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} + \frac{\mu_0}{4\pi}\int d^3\vec{r}\,'\frac{\vec{J}_M(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} + \frac{\mu_0}{4\pi}\int dS'\frac{\vec{K}_M(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}} \quad (A)$$

c.f. $$\varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3\vec{r}\,'\frac{\rho_f(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} + \frac{1}{4\pi\epsilon_0}\int d^3\vec{r}\,'\frac{\rho_P(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} + \frac{1}{4\pi\epsilon_0}\int dS'\frac{\sigma_P(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

the form of (A') immediately suggests that the $\vec{B}$-field will follow from a modification to the Biot-Savart law to include the magnetization currents

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3\vec{r}\,' \left(\vec{J}(\vec{r}\,') + \vec{J}_M(\vec{r}\,')\right) \times \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} + \frac{\mu_0}{4\pi}\int dS' \, \vec{K}_M(\vec{r}\,') \times \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

the nature of the surface current $\vec{K}_M \equiv \vec{M}\times\hat{n}$ can be explored using the simple example of a uniform magnetization in a cuboid, $\vec{M} = M_0\hat{z}$

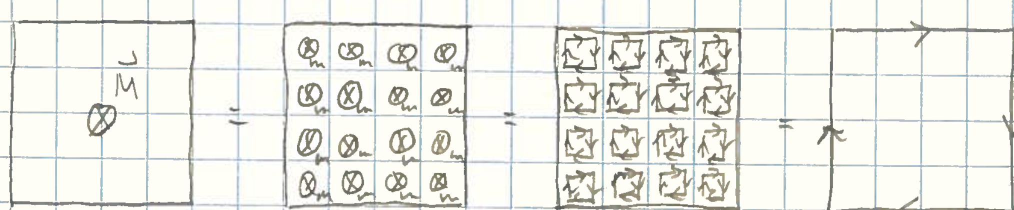


top face, $\hat{n} = \hat{x}$
$\vec{K}_M = M_0\,\hat{z}\times\hat{x} = M_0\hat{y}$

right side face $\hat{n} = \hat{y}$
$\vec{K}_M = M_0\,\hat{z}\times\hat{y} = -M_0\hat{x}$

So the surface current flows around the outer surface as shown

we can understand this in terms of (fictitious) current loops giving the magnetic dipole moments which sum up to the magnetisation



returning to eqn (A) with the integrals now over all space, if the magnetization is localized we can write

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3\vec{r}\,' \frac{\vec{J}(\vec{r}\,') + \vec{\nabla}_{r'}\times\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

and it follows that $\vec{\nabla}\times\vec{B} = \mu_0(\vec{J} + \vec{\nabla}\times\vec{M})$

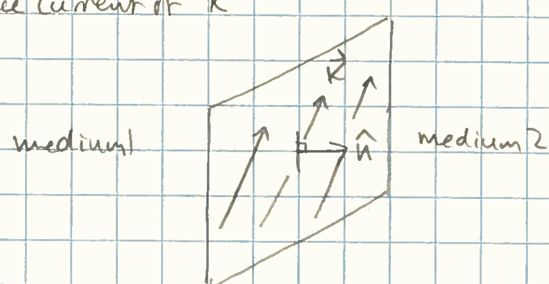
$$\Rightarrow \quad \vec{\nabla}\times\left(\frac{1}{\mu_0}\vec{B} - \vec{M}\right) = \vec{J}$$

and we can define a useful auxillary field, $\underline{\vec{H} \equiv \frac{1}{\mu_0}\vec{B} - \vec{M}}$ $\quad \left(\vec{B} = \mu_0(\vec{H}+\vec{M})\right)$

(which is in some ways similar to $\vec{D} = \varepsilon_0\vec{E} + \vec{P}$ when dealing with dielectrics)

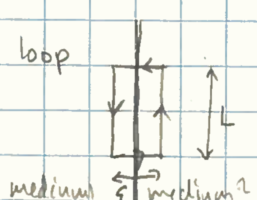
the equations satisfied by $\vec{H}, \vec{B}$ are $\boxed{\begin{aligned}\vec{\nabla}\times\vec{H} &= \vec{J}_f \\ \vec{\nabla}\cdot\vec{B} &= 0\end{aligned}}$ $\quad \left[\begin{aligned}&\text{compare} \\ &\vec{\nabla}\cdot\vec{D} = \rho_f \\ &\vec{\nabla}\times\vec{E} = 0\end{aligned}\right]$

At a boundary between two magnetizable media, where there may be a <u>free</u> surface current of $\vec{K}$



- a Gaussian pillbox straddling the boundary $\Rightarrow \vec{B}_2\cdot\hat{n} - \vec{B}_1\cdot\hat{n} = 0 \quad \Rightarrow \underline{(\vec{B}_2 - \vec{B}_1)\cdot\hat{n} = 0}$

- a closed loop

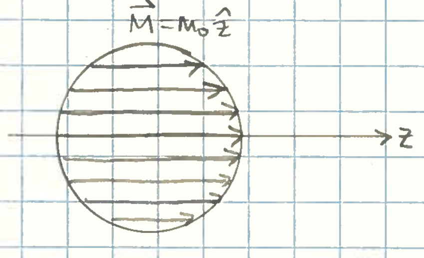


$$\Rightarrow \oint \vec{H}\cdot d\vec{\ell} = I_f \quad \Rightarrow (H_2^{\parallel} - H_1^{\parallel})\,L = I_f$$

$$\Rightarrow \underline{\hat{n}\times(\vec{H}_2 - \vec{H}_1) = \vec{K}_f}$$

A uniformly magnetized sphere

Suppose we have a sphere of radius R, centered at the origin, containing uniform magnetization $\vec{M}$ that we'll choose to lie along the z-axis

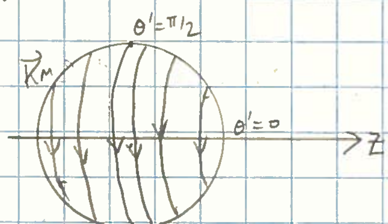


Let's try to find the magnetic field, $\vec{B}$, everywhere due to this object. One approach is to use eqn A' on page 132 to find the vector potential

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3\vec{r}\,'\, \frac{\vec{\nabla}_{r'}\times\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} + \frac{\mu_0}{4\pi}\int dS'\,\frac{\vec{M}(\vec{r}\,')\times\hat{n}'}{|\vec{r}-\vec{r}\,'|}$$

for uniform magnetization, $\vec{\nabla}\times\vec{M}=0$ & we have only the surface term

$$\vec{K}_M = \vec{M}(\vec{r}\,')\times\hat{n}' = M_0\,\hat{z}\times\hat{r} = M_0\left(\hat{r}\cos\theta' - \hat{\theta}\sin\theta'\right)\times\hat{r} = -M_0\sin\theta'\,\hat{\theta}\times\hat{r} = M_0\sin\theta'\,\hat{\phi}$$



$$A(\vec{r}) = \frac{\mu_0 M_0}{4\pi}R^2\int d\Omega'\,\frac{\sin\theta'\,\hat{\phi}}{|\vec{r}-\vec{r}\,'|}$$

$$\hat{\phi} = -\sin\phi'\,\hat{x} + \cos\phi'\,\hat{y}$$

$$\& \quad \frac{1}{|\vec{r}-\vec{r}\,'|} = 4\pi\sum_{\ell m}\frac{1}{2\ell+1}\frac{r_<^{\ell}}{r_>^{\ell+1}}Y^*_{\ell m}(\theta'\phi')\,Y_{\ell m}(\theta\phi)$$

$r_< = \min(r, r')$
$r_> = \max(r, r')$

$$\sin\theta'\,\hat{\phi} = -\sin\theta'\sin\phi'\,\hat{x} + \sin\theta'\cos\phi'\,\hat{y}$$

$$= -\frac{1}{2i}\left(\sin\theta' e^{i\phi'} - \sin\theta' e^{-i\phi'}\right)\hat{x} + \frac{1}{2}\left(\sin\theta' e^{i\phi'} + \sin\theta' e^{-i\phi'}\right)\hat{y}$$

$$= \frac{1}{2}\sin\theta' e^{i\phi'}\left(i\hat{x}+\hat{y}\right) + \frac{1}{2}\sin\theta' e^{-i\phi'}\left(-i\hat{x}+\hat{y}\right)$$

$$= -\sqrt{\frac{2\pi}{3}}\,Y_{1+1}(\theta'\phi')\left(i\hat{x}+\hat{y}\right) + \sqrt{\frac{2\pi}{3}}\,Y_{1-1}(\theta'\phi')\left(-i\hat{x}+\hat{y}\right)$$

using $\int d\Omega'\,Y^*_{\ell m}(\theta'\phi')\,Y_{\ell' m'}(\theta'\phi') = \delta_{\ell\ell'}\delta_{mm'}$

$$\vec{A}(\vec{r}) = \frac{\mu_0 M_0}{3}\frac{R^2 r_<}{r_>^2}\left(-\sqrt{\frac{2\pi}{3}}\,Y_{1+1}(\theta,\phi)\left(i\hat{x}+\hat{y}\right) + \sqrt{\frac{2\pi}{3}}\,Y_{1-1}(\theta,\phi)\left(-i\hat{x}\right.\right.$$

$$\Rightarrow \vec{A}(\vec{r}) = \frac{\mu_0 M_0}{3} R^2 \frac{r_<}{r_>^2} \sin\theta\,\hat{\phi} = \begin{cases} \frac{\mu_0 M_0}{3} r\sin\theta\,\hat{\phi} & r<R \\ \frac{\mu_0 M_0}{3} \frac{R^3}{r^2} \sin\theta\,\hat{\phi} & r>R \end{cases}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \hat{r}\,\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \hat{\theta}\,\frac{1}{r}\frac{\partial}{\partial r}(r A_\phi)$$

* inside the sphere $(r<R)$ $\quad \vec{B} = \frac{2\mu_0}{3} M_0 \left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right) = \frac{2\mu_0}{3} M_0\,\hat{z}$

$$\boxed{\vec{B} = \frac{2}{3}\mu_0 \vec{M}}$$

a uniform field parallel to $\vec{M}$

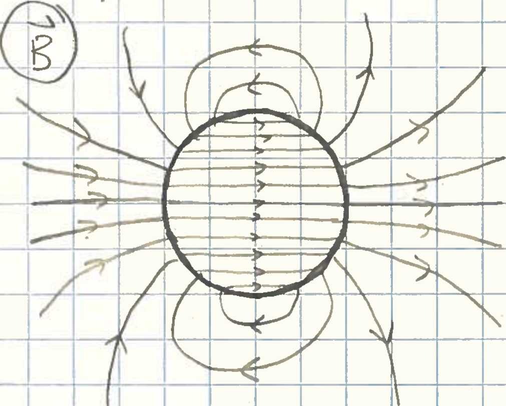
* outside the sphere $(r>R)$ $\quad \vec{B} = \frac{\mu_0 M_0}{3}\frac{R^3}{r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{3\hat{r}\,\vec{m}\cdot\hat{r} - \vec{m}}{r^3}\right] \quad \text{if} \quad \vec{m} = \frac{4}{3}\pi R^3 \vec{M}}$$

i.e the field from a magnetic dipole of moment $\vec{m}$.

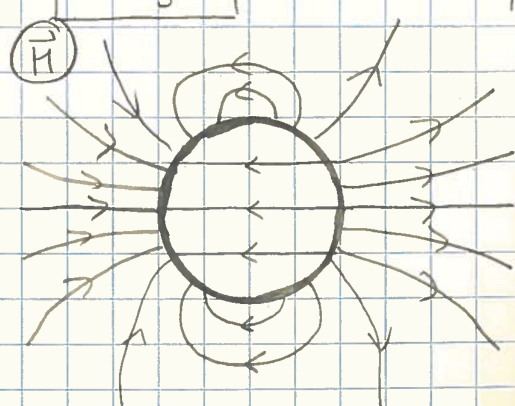
$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$ so inside the sphere $\boxed{\vec{H} = -\frac{1}{3}\vec{M}}$ & outside $\vec{H} = \frac{1}{\mu_0}\vec{B}$

$\vec{B}$



$\vec{B}$ continuous at the surface

$\vec{H}$



lines of $\vec{H}$ can start & end on the surface

there's another approach we can use in cases like these where $\vec{J} = \vec{0}$.
In this case $\vec{\nabla} \times \vec{H} = 0 \Rightarrow \vec{H}$ can be written in terms of a scalar potential, $\vec{H} = -\vec{\nabla}\phi_M$.

The "other" magnetic eqn $\vec{\nabla}\cdot\vec{B} = 0$ & $\vec{B} = \mu_0(\vec{H} + \vec{M}) \Rightarrow \vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M}$
$\Rightarrow \nabla^2\phi_M = \vec{\nabla}\cdot\vec{M}$

and the structure of this eqn is the same as the electrostatics Poisson eqn
if we interpret $-\vec{\nabla}\cdot\vec{M}$ as an "effective magnetic charge density", ρ_M,
and we can transcribe our electrostatic methods: [see page 71]

$$\phi_M(\vec{r}) = \frac{1}{4\pi}\int d^3\vec{r}\,'\,\rho_M(\vec{r}\,')\frac{1}{|\vec{r}-\vec{r}\,'|} + \frac{1}{4\pi}\int dS'\,\frac{\sigma_M(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

$$\rho_M = -\vec{\nabla}\cdot\vec{M}$$
$$\sigma_M = \hat{n}\cdot\vec{M}$$

our uniformly magnetized sphere is analogous to the uniformly polarized sphere (page 53 & 71) - you can check that we'll get the same results as on page 138.

MAGNETIC MEDIA

Materials show a wider variety of magnetic behaviors than they do dielectric behavior.

A common behavior is known as DIAMAGNETISM where a sample responds to an applied field with a magnetisation in the opposite direction.

defining the magnetic susceptibility χ_m in $\vec{M} = \chi_m \vec{H}$, diagmagnetism corresponds to $\chi_m < 0$.

Other materials are observed to exhibit PARAMAGNETISM, where the induced magnetisation is parallel to the applied field and $\chi_m > 0$.

The most extreme magnetic behavior is known as FERROMAGNETISM, in which we observe very lage values of χ_m that are strongly dependent on the applied field. "Hard" ferromagnets can even possess a magnetization in the absence of any applied field - they can be "permanent" magnets.

If we define a "permeability" μ in $\vec{B} = \mu \vec{H}$ then $\mu \vec{H} = \mu_0(\vec{H} + \vec{M}) = \mu_0 \vec{H} + \mu_0 \chi_m \vec{H}$

& $\underline{\mu = \mu_0(\chi_m + 1)}$

$(\mu/\mu_0)_{dia} \simeq 1 - \varepsilon$ / $(\mu/\mu_0)_{para} \simeq 1 + \varepsilon$ / μ_{ferro} is not a simple constant.

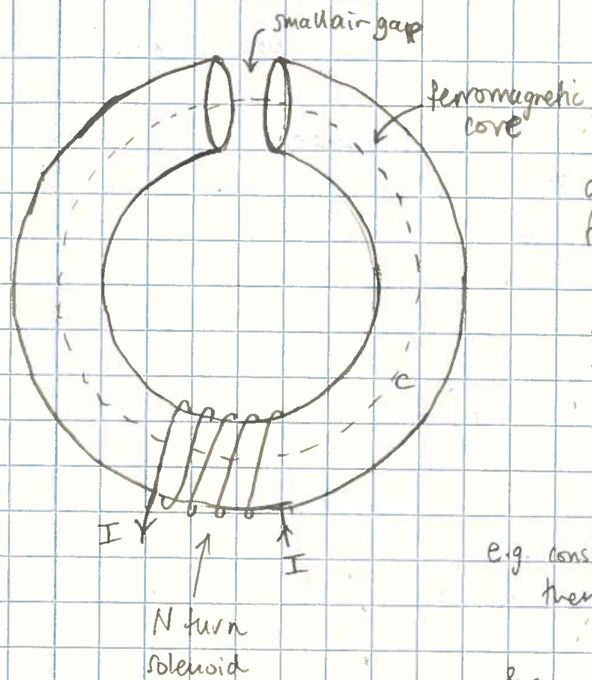
e.g. uniformly magnetized sphere of diamagnetic or paramagnetic material in a uniform external field

$$\vec{B} = \vec{B}_{ext} + \frac{2}{3}\mu_0 \vec{M} = \mu \vec{H} = \mu\Big(\underbrace{\vec{H}_{ext}}_{\vec{B}_{ext}/\mu_0} - \frac{1}{3}\vec{M}\Big)$$

$$\vec{B}_{ext} + \frac{2}{3}\mu_0 \vec{M} = \mu/\mu_0 \, \vec{B}_{ext} - \frac{1}{3}\mu \vec{M}$$

$$\boxed{\vec{M} = 3\,\frac{\mu/\mu_0 - 1}{\mu/\mu_0 + 2}\,\vec{B}_{ext}}$$

FERROMAGNETIC HYSTERESIS



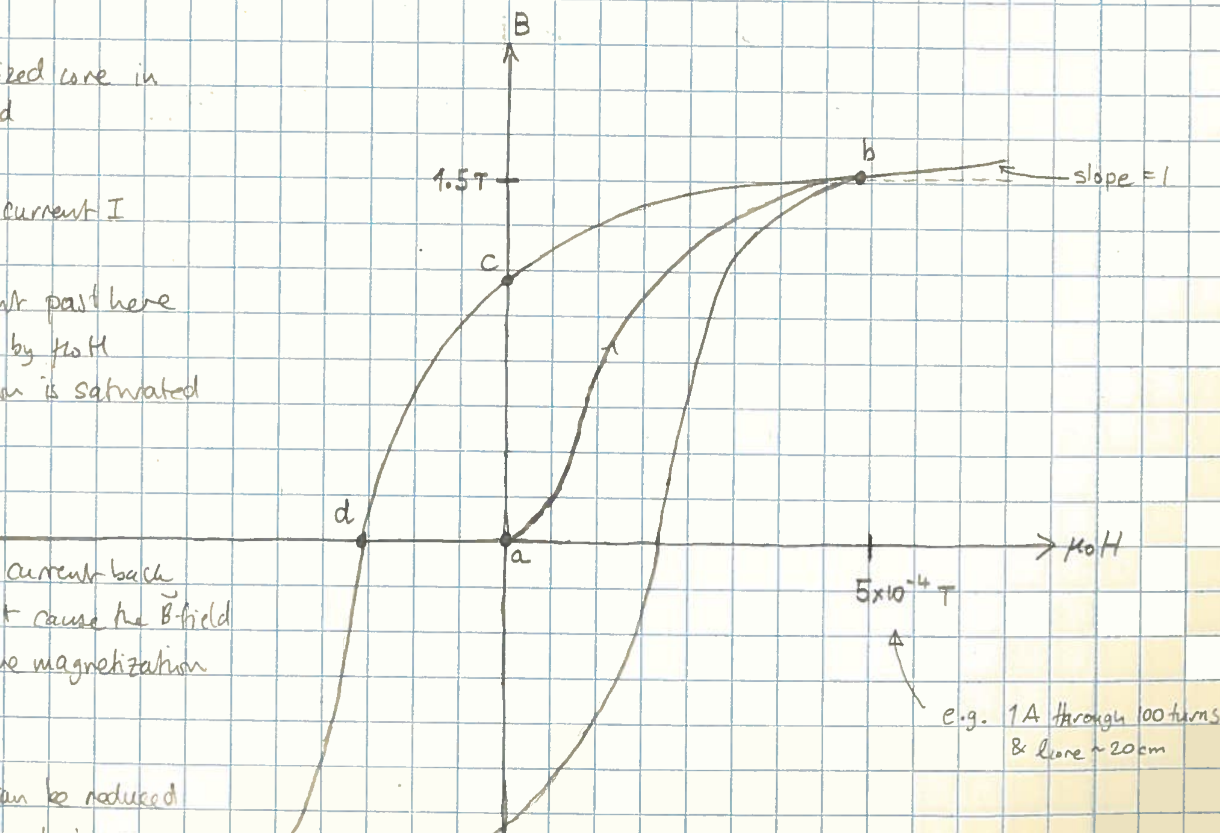
assume all the flux remains in the core, apart from that in the small air gap

$$\oint_C \vec{H}\cdot d\vec{l} = NI = H_{core}\, l_{core} + H_{gap}\, l_{gap}$$

$$\& \quad B_{core} = B_{gap}$$

e.g. consider the case in which there is no air gap, then $H = NI/l_{core}$

& suppose we examine the magnetic field B as we vary the current I (and thus the value of H)



usly unmagnetized core in
applied field

increase the current I

reasing current past here
increases B by $\mu_0 H$
magnetization is saturated

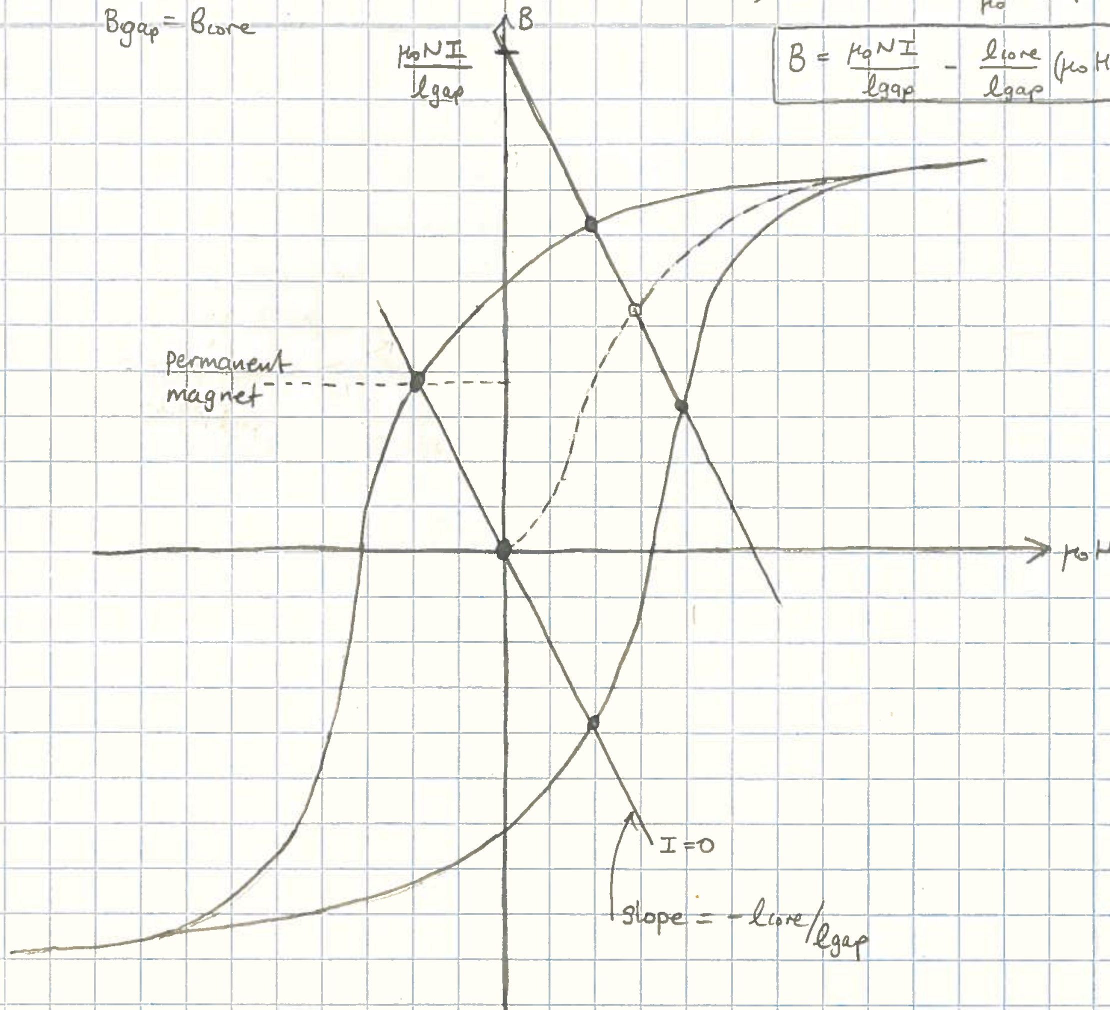
decreasing the current back
zero does not cause the $\vec{B}$-field
vanish - some magnetization
mains.

the $\vec{B}$-field can be reduced
zero only by applying an
ssing field.

electromagnet $B_{gap} = \mu_0 H_{gap}$ & $H_{core}\, l_{core} + H_{gap}\, l_{gap} = NI$ $\Rightarrow$ $H_{core}\, l_{core} + \frac{B_{core}}{\mu_0} l_{gap} = NI$

$B_{gap} = B_{core}$

$$\boxed{B = \frac{\mu_0 N I}{l_{gap}} - \frac{l_{core}}{l_{gap}}(\mu_0 H)}$$



a sphere in an external field

$$\left.\begin{aligned} \vec{B} &= \vec{B}_{ext} + \tfrac{2}{3}\mu_0 \vec{M} \\ \vec{H} &= \tfrac{1}{\mu_0}\vec{B}_{ext} - \tfrac{1}{3}\vec{M} \end{aligned}\right\} \vec{B} + 2\mu_0 \vec{H} = \vec{B}_{ext} + 2\vec{B}_{ext}$$

$\Rightarrow$ $\underline{B = 3B_{ext} - 2(\mu_0 H)}$

also straight lines in diagram above.

MAGNETIC SHIELDING

Suppose we place a hollow sphere (inner radius a, outer radius b) of magnetic material in a uniform magnetic field. We'll consider the case that the material is diamagnetic, paramagnetic or is a "soft" ferromagnet, so we can write $\vec{B} = \mu \vec{H}$ with constant μ inside the material.

Since there are no free currents in the problem we have $\vec{\nabla} \times \vec{H} = 0$ & we can use a magnetic scalar potential, $\vec{H} = -\vec{\nabla}\phi_M$.

Since $\vec{B} = \mu\vec{H}$, $\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot\vec{H} = 0$ & thus $\underline{\nabla^2 \phi_M = 0}$

At the boundaries between vacuum & material we have H_θ continuous & B_r continuous

i.e. $\left.\dfrac{\partial \phi_M}{\partial \theta}\right|_{I} = \left.\dfrac{\partial \phi_M}{\partial \theta}\right|_{II}$ & $\mu_I \left.\dfrac{\partial \phi_M}{\partial r}\right|_{I} = \mu_{II}\left.\dfrac{\partial \phi_M}{\partial r}\right|_{II}$

Outside the sphere $(r>b)$ we have $\phi_M = -H_0 r\cos\theta + \sum_{\ell=0}^{\infty} \alpha_\ell \dfrac{1}{r^{\ell+1}} P_\ell(\cos\theta)$

in the magnetic medium $(a<r<b)$ $\phi_M = \sum_{\ell=0}^{\infty}\left(\beta_\ell r^\ell + \gamma_\ell \dfrac{1}{r^{\ell+1}}\right) P_\ell(\cos\theta)$

in the spherical cavity $(r<a)$ $\phi_M = \sum_{\ell=0}^{\infty} \delta_\ell r^\ell P_\ell(\cos\theta)$

applying the matching conditions we obtain the simultaneous eqns

$$
\begin{array}{rrrrcl}
\alpha_1 & -b^3\beta_1 & -\gamma_1 & & = & b^3 H_0 \\
2\alpha_1 & +\left(\frac{\mu}{\mu_0}\right) b^3 \beta_1 & -2\left(\frac{\mu}{\mu_0}\right)\gamma_1 & & = & -b^3 H_0 \\
 & a^3 \beta_1 & +\gamma_1 & -a^3\delta_1 & = & 0 \\
 & \left(\mu/\mu_0\right) a^3 \beta_1 & -\left(\mu/\mu_0\right)\gamma_1 & -a^3\delta_1 & = & 0
\end{array}
$$

and their solution includes

$$\delta_1 = \frac{-9\mu\mu_0 H_0}{(2\mu+\mu_0)(\mu+2\mu_0) - 2\left(a^3/b^3\right)(\mu-\mu_0)^2}$$

the uniform field in the cavity is reduced from H_0 for both diamagnets and paramagnets. In the case of ferromagnets with $\mu \gg \mu_0$, $\delta_1 \longrightarrow -\dfrac{9\mu_0}{2\mu}\dfrac{1}{1-a^3/b^3} H_0$

which may be a rather large reduction in field.

e.g. "mu-metal" (nickel, iron, molybdenum ...)