MAGNETOSTATICS "$\vec{\nabla}\cdot\vec{J} = 0$"

There are several physical phenomena which hint at the existance of another field which interacts with electrical charge.

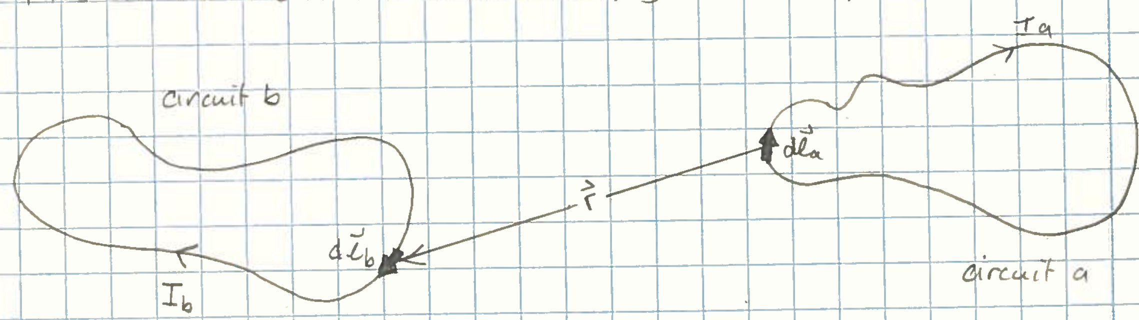
For example, when placed near a large bar of iron, or a wire carrying an electric current we may observe:

- → a torque on a small bar of iron (1)
- → a force on a current carrying wire (2)
- → deflection of the path of an electrically charged particle (3)

In each case, the effect proves to be due to the presence of a "magnetic field" $\vec{B}$

(1) $\vec{\tau} = \vec{\mu}\times\vec{B}$ (2) $d\vec{F} = I d\vec{\ell}\times\vec{B}$ (3) $\vec{F} = q\vec{v}\times\vec{B}$

Let's focus on the case of two circuits carrying currents separated in space



empirically we find that the force on b due to a is given by

$$\vec{F} = \frac{\mu_0}{4\pi} I_a I_b \oint_b d\vec{\ell}_b \times \oint_a d\vec{\ell}_a \times \frac{\hat{r}}{r^2} \qquad \text{with } \mu_0 = 4\pi\times10^{-7}\ \text{N/A}^2$$

and we may identify the effect of a at the remote location of b by the field

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I_a \oint_a d\vec{\ell}_a \times \frac{\hat{r}}{r^2}$$

"law of Biot & Savart"

we can generalize to cases of current density other than simple wires $I_a \oint d\vec{\ell}_a \to \int d^3\vec{r}\,' \vec{J}(\vec{r}\,')$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3\vec{r}\,'\ \vec{J}(\vec{r}\,') \times \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3\vec{r}\,' \, \vec{J}(\vec{r}\,') \times \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} \quad (B)$$

compare with $\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3\vec{r}\,' \rho(\vec{r}\,') \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$

a region of current density placed in this field will feel a force

$$\vec{F} = \int d^3\vec{r} \, \vec{J}(\vec{r}) \times \vec{B}(\vec{r})$$

and a torque $\vec{\tau} = \int d^3\vec{r} \; \vec{r} \times \left(\vec{J}(\vec{r}) \times \vec{B}(\vec{r})\right)$

We can obtain a different form of eqn (B) using the result $\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} = -\vec{\nabla}_r \frac{1}{|\vec{r}-\vec{r}\,'|}$

then $\vec{B}(\vec{r}) = -\frac{\mu_0}{4\pi}\int d^3\vec{r}\,' \, \vec{J}(\vec{r}\,') \times \vec{\nabla}_r \frac{1}{|\vec{r}-\vec{r}\,'|}$

since $\vec{\nabla} \times (f\vec{J}) = f\vec{\nabla}\times\vec{J} - \vec{J}\times\vec{\nabla}f$

& $\vec{\nabla}_r \vec{J}(\vec{r}\,') = 0$

$\Rightarrow \vec{J}\times\vec{\nabla}f = -\vec{\nabla}\times(f\vec{J})$

$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\vec{\nabla}\times\int d^3\vec{r}\,' \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}}$$

it immediately follows that $\underline{\vec{\nabla}\cdot\vec{B} = 0}$

if we define $\vec{A} = \frac{\mu_0}{4\pi}\int d^3\vec{r}\,' \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$ then $\vec{B} = \vec{\nabla}\times\vec{A}$

& $\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

$$= \frac{\mu_0}{4\pi}\vec{\nabla}\int d^3\vec{r}\,' \, \vec{J}(\vec{r}\,') \cdot \underbrace{\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}\,'|}}_{-\vec{\nabla}_{r'}\frac{1}{|\vec{r}-\vec{r}\,'|}} - \frac{\mu_0}{4\pi}\int d^3\vec{r}\,' \, \vec{J}(\vec{r}\,') \underbrace{\nabla^2\frac{1}{|\vec{r}-\vec{r}\,'|}}_{-4\pi\delta(\vec{r}-\vec{r}\,')}$$

$$= \frac{\mu_0}{4\pi}\vec{\nabla}\int d^3\vec{r}\,' \frac{\vec{\nabla}_{r'}\cdot\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} + \mu_0\vec{J}(\vec{r}) \Rightarrow \boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}}$$

$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$ "Ampere's law"

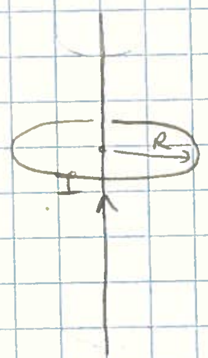
Consider an integral over an open surface

$$\underbrace{\int_S d\vec{S}\cdot\vec{\nabla}\times\vec{B}}_{\text{by Stokes' law}} = \mu_0 \int_S d\vec{S}\cdot\vec{J} = \mu_0 I \quad , \; I = \text{current through the surface, } S$$

$$= \oint_C d\vec{\ell}\cdot\vec{B} \quad \text{where the path } C \text{ is the boundary of the surface } S$$

$$\boxed{\oint_C d\vec{\ell}\cdot\vec{B} = \mu_0 I} \quad \text{to be compared to} \quad \int_S d\vec{S}\cdot\vec{E} = Q/\epsilon_0$$

e.g. consider a long straight wire lying along the z-axis

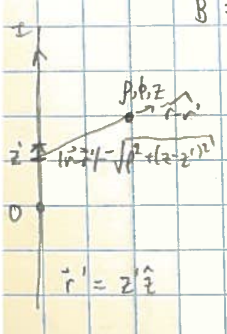


an appropriate C is a circle of radius R centered at the wire,

assuming $\vec{B} = B_\phi \hat{\phi}$ (work out how we can reject $\hat{z}$ & $\hat{\rho}$ terms)

we have $B_\phi \cdot 2\pi R = \mu_0 I$ $\Rightarrow \underline{\vec{B} = \dfrac{\mu_0 I}{2\pi\rho}\hat{\phi}}$

for an alternative derivation we can use the Biot-Savart law (treating the remainder of the circuit as being at infinity)



$$\vec{B} = \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty} dz' \, \frac{\hat{z}\times(\rho\hat{\rho} + z\hat{z} - z'\hat{z})}{(\rho^2+(z-z')^2)^{3/2}} = \frac{\mu_0 I}{4\pi}\hat{\phi}\rho\int_{-\infty}^{\infty}\frac{dz'}{(\rho^2+(z-z')^2)^{3/2}} = \frac{\mu_0 I}{4\pi}\frac{1}{\rho}\hat{\phi}\int_{-\infty}^{\infty}\frac{d\xi}{(1+\xi^2)^{3/2}}$$

$$\xi = \frac{z-z'}{\rho}$$

$$\left[\begin{array}{l} \xi = \tan\theta \\ d\xi = d\theta/\cos^2\theta \\ 1+\tan^2\theta = 1/\cos^2\theta \\ \int_{-\pi/2}^{\pi/2}\cos\theta\, d\theta = 2 \end{array}\right.$$

$$= \frac{\mu_0 I}{2\pi\rho}\hat{\phi}$$

AN ALTERNATIVE FORM OF THE BIOT-SAVART LAW FOR A CURRENT LOOP

Starting with $d\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} d\vec{l}' \times \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$ we can show that the field from a closed loop of current at position $\vec{r}$ can be written $\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \vec{\nabla}\Omega$

where Ω is the solid angle subtended by the loop at position $\vec{r}$.

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consider for example the x-component, integrated over the loop

$$X = \hat{x} \cdot \left[\oint d\vec{l}' \times \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \right]$$

we already know that $\vec{\nabla}_{\vec{r}'} \frac{1}{|\vec{r}-\vec{r}'|} = \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$

$$X = \hat{x} \cdot \oint d\vec{l}' \times \vec{\nabla}_{\vec{r}'}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = \oint d\vec{l}' \cdot \vec{\nabla}_{\vec{r}'}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) \times \hat{x}$$

using cylic property of triple scalar product

STOKES THEOREM: $\oint d\vec{l} \cdot \vec{A} = \int d\vec{S} \cdot \vec{\nabla} \times \vec{A}$

$$X = \int d\vec{S}' \cdot \vec{\nabla}_{\vec{r}'} \times \left[\vec{\nabla}_{\vec{r}'}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) \times \hat{x} \right]$$

$$\vec{\nabla} \times (\vec{f} \times \vec{g}) = \vec{f}(\vec{\nabla}\cdot\vec{g}) - \vec{g}(\vec{\nabla}\cdot\vec{f}) - (\vec{f}\cdot\vec{\nabla})\vec{g} + (\vec{g}\cdot\vec{\nabla})\vec{f}$$

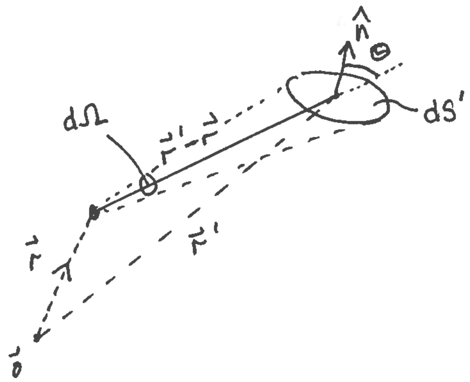
$$= \int d\vec{S}' \cdot \left[\underbrace{-\vec{\nabla}^2_{\vec{r}'}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)}_{4\pi\delta^{(3)}(\vec{r}-\vec{r}')} \hat{x} + \hat{x}\cdot\vec{\nabla}_{\vec{r}'} \, \vec{\nabla}_{\vec{r}'} \frac{1}{|\vec{r}-\vec{r}'|} \right]$$

$\rightarrow$ if $\vec{r}$ is not on the surface of integration (and we can always choose $\vec{S}$ so that it isn't) this contributes nothing

$$= \int dS' \, \hat{x}\cdot\vec{\nabla}_{\vec{r}'} \, \hat{n}\cdot\vec{\nabla}_{\vec{r}'}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = \int dS' \, \hat{x}\cdot\vec{\nabla}_{\vec{r}'} \underbrace{\left(\frac{\hat{n}\cdot(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right)}_{\text{antisym in } \vec{r} \leftrightarrow \vec{r}'}$$

$\Rightarrow \vec{\nabla}_{\vec{r}'} f(\vec{r},\vec{r}') = -\vec{\nabla}_{\vec{r}} f(\vec{r},\vec{r}')$

$$= -\hat{x}\cdot\vec{\nabla}_{\vec{r}} \int dS' \frac{\hat{n}\cdot(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$



$$d\Omega = \frac{dS'\cos\theta}{|\vec{r}-\vec{r}\,'|^2} = \frac{dS'}{|\vec{r}-\vec{r}\,'|^2}\,\frac{\hat{n}\cdot(\vec{r}\,'-\vec{r})}{|\vec{r}\,'-\vec{r}|}$$

$$= -dS'\,\frac{\hat{n}\cdot(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

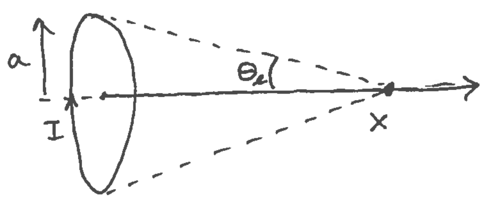
so $X = \hat{x}\cdot\vec{\nabla}\int d\Omega = \underline{\hat{x}\cdot\vec{\nabla}\Omega}$

repeating this for the other two cartesian unit vectors

$$\Rightarrow \quad \underline{\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\vec{\nabla}\Omega}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \vec{\nabla}\Omega$$

e.g. "easy" way to find the field on the symmetry axis from a circular current loop



$$\Omega = \int_0^{2\pi} d\phi \int_0^{\theta_\ell} \sin\theta \, d\theta = 2\pi \cos\theta_\ell$$

$$\Omega = 2\pi \cdot \frac{x}{\sqrt{x^2+a^2}}$$

$$\vec{\nabla}\Omega = \hat{x} \frac{d}{dx}\left[2\pi \frac{x}{\sqrt{x^2+a^2}} \right] = \hat{x} \cdot 2\pi \cdot \frac{a^2}{(x^2+a^2)^{3/2}}$$

$$\& \quad \vec{B}(x) = \hat{x} \frac{\mu_0 \cdot \frac{1}{2} I a^2}{(x^2+a^2)^{3/2}}$$

The Vector Potential

We've found that in magnetostatic situations, $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ & $\vec{\nabla}\cdot\vec{B} = 0$

$\vec{\nabla}\cdot\vec{B} = 0$ suggests that we can write $\vec{B} = \vec{\nabla}\times\vec{A}$ for some suitable $\vec{A}$. In fact we've already seen how to do this in a way compatible with $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3\vec{r}\,'\,\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}.$$

But actually we could add the gradient of any function to this and still satisfy both equations for $\vec{B}$:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3\vec{r}\,'\,\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} + \vec{\nabla}\Psi$$

addition of this is called a "gauge transformation"

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J} \quad\Rightarrow\quad \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J}$$

and part of the freedom to make gauge transformations is removed if we make a choice like

$\vec{\nabla}\cdot\vec{A} = 0$ the "Coulomb gauge" choice

then $\underline{\nabla^2\vec{A} = -\mu_0\vec{J}}$ $\Rightarrow$ $\nabla^2 A_i = -\mu_0 J_i$ three Poisson's equations

some gauge freedom is still present, we can add the gradient of any function satisfying $\nabla^2\Psi = 0$. If we demand that the fields fall to zero at infinity, Ψ can only be an irrelevant constant.

in Coulomb gauge in an unbounded space

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3\vec{r}\,'\,\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

* A brief mathematical aside to prove two useful identities

Consider two scalar functions $f(\vec{r})$, $g(\vec{r})$ and a vector function $\vec{A}(\vec{r})$

$$\int d^3\vec{r}\ \vec{\nabla}\cdot(fg\vec{A}) = \int d\vec{S}\cdot\vec{A}fg = 0$$ if the functions fall to zero at infinity (e.g. suppose $\vec{A}$ is localized)

surface at infinity

$$0 = \int d^3\vec{r}\left[fg\,\vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}(fg)\right] = \int d^3\vec{r}\left[fg\,\vec{\nabla}\cdot\vec{A} + f\,\vec{A}\cdot\vec{\nabla}g + g\,\vec{A}\cdot\vec{\nabla}f\right]$$

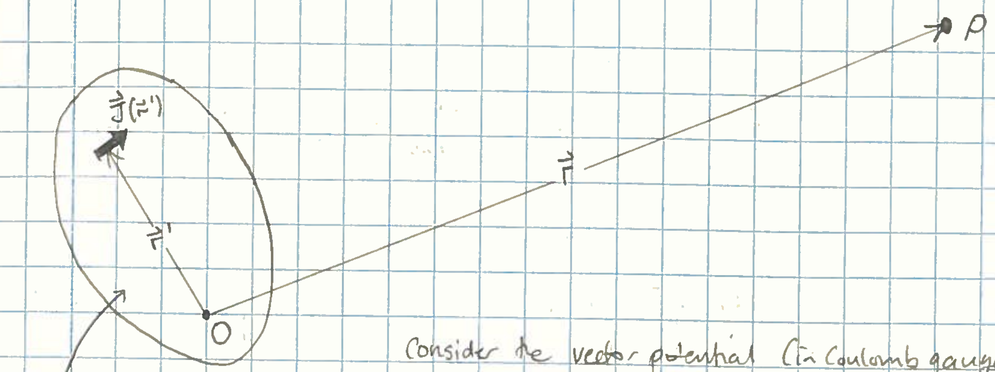
→ $f(\vec{r}) = 1$, $g(\vec{r}) = r_i$, and $\vec{\nabla}\cdot\vec{A} = 0$

then $\underline{0 = \int d^3\vec{r}\ A_i}$. (I)

→ $f(\vec{r}) = r_i$, $g(\vec{r}) = r_j$, and $\vec{\nabla}\cdot\vec{A} = 0$

then $\underline{0 = \int d^3\vec{r}\left(r_i A_j + r_j A_i\right)}$. (II)

$\boxed{\text{The field far from an arbitrary localized distribution of current}}$



current only in this region, V_J

Consider the vector potential (in Coulomb gauge) at P

$$A_i(\vec{r}) = \frac{\mu_0}{4\pi}\int_{V_J} d^3\vec{r}\,' \frac{J_i(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} = \frac{\mu_0}{4\pi}\int d^3\vec{r}\,' \frac{J_i(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

over all space since $\vec{J}$ is localized to V_J

now provided $|\vec{r}| \gg |\vec{r}\,'|$, $\quad \dfrac{1}{|\vec{r}-\vec{r}\,'|} = \dfrac{1}{r} + \dfrac{\vec{r}\cdot\vec{r}\,'}{r^3} + O(r'^2)$

(this is a basic multipole expansion)

then $$A_i(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r}\underbrace{\int d^3\vec{r}\,' J_i(\vec{r}\,')}_{=0 \text{ by (I)}} + \frac{\mu_0}{4\pi}\frac{r_j}{r^3}\underbrace{\int d^3 r' \, r_j' J_i(\vec{r}\,')}_{\frac{1}{2}\int d^3\vec{r}\,'\left(r_j' J_i - r_i' J_j\right) \text{ by (II)}} + \ldots$$

$$A_i(\vec{r}) = \frac{1}{2}\cdot\frac{1}{r^3}\frac{\mu_0}{4\pi} r_j \int d^3\vec{r}\,' \left(r_j' J_i - r_i' J_j\right) + \ldots$$

consider $$\left[\vec{r}\times(\vec{r}\,'\times\vec{J})\right]_i = \epsilon_{ijk}\, r_j\, (\vec{r}\,'\times\vec{J})_k = \epsilon_{ijk}\, r_j\, \epsilon_{klm}\, r_l'\, J_m$$

$$= \epsilon_{ijk}\epsilon_{klm}\, r_j r_l' J_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, r_j r_l' J_m$$

$$= r_i'(r_j J_j) - J_i(r_j r_j')$$

so $$\vec{A}(\vec{r}) = -\frac{1}{2}\frac{1}{r^3}\frac{\mu_0}{4\pi}\vec{r}\times\int d^3\vec{r}\,' \,\vec{r}\,'\times\vec{J}(\vec{r}\,') + \ldots$$

$$\underline{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3}}$$ with $\vec{m} = \frac{1}{2}\int d^3\vec{r}\,' \,\vec{r}\,'\times\vec{J}(\vec{r}\,')$ the "magnetic moment"

the magnetic field follows from the curl of this expression

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3} = \frac{\mu_0}{4\pi} \vec{g}(\vec{r}) \times \vec{r} \qquad \text{with } \vec{g} = \vec{m}/r^3 \qquad \text{where } \vec{m} \text{ is a constant}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{\mu_0}{4\pi} \vec{\nabla} \times (\vec{g} \times \vec{r}) = \frac{\mu_0}{4\pi} \left[2\vec{g} + r \frac{\partial}{\partial r} \vec{g} - \vec{r}\, \vec{\nabla} \cdot \vec{g} \right]$$

$$\vec{\nabla} \cdot \vec{g} = \vec{m} \cdot \vec{\nabla} \frac{1}{r^3} = -3\, \vec{m} \cdot \hat{r} / r^4$$

$$= \frac{\mu_0}{4\pi} \left[\frac{2\vec{m}}{r^3} - \frac{3\vec{m}}{r^3} + 3\vec{r}\, \frac{\vec{m} \cdot \hat{r}}{r^4} \right]$$

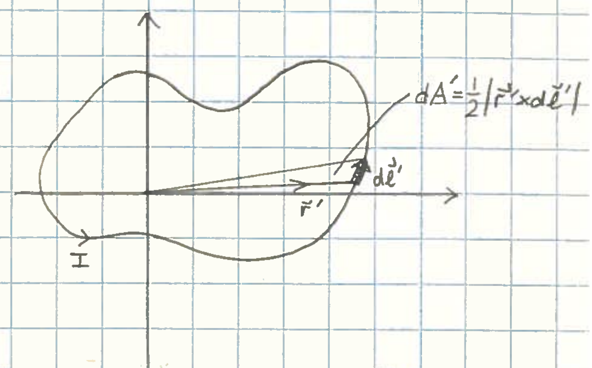
$$\boxed{\vec{B} = \frac{\mu_0}{4\pi} \left[\frac{3\hat{r}\, \vec{m} \cdot \hat{r} - \vec{m}}{r^3} \right]}$$

c.f. the $\vec{E}$ field from an electric dipole, $\vec{p}$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \left[\frac{3\hat{r}\, \vec{p} \cdot \hat{r} - \vec{p}}{r^3} \right]$$

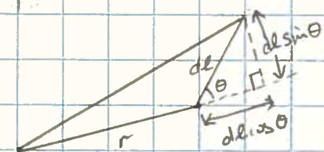
an important example case is a closed circuit of current carrying wire in a plane,

then $\vec{m} = \frac{1}{2}\int d^3\vec{r}\,'\ \vec{r}\,' \times \vec{J}(\vec{r}\,') = \frac{1}{2} I \int \vec{r}\,' \times d\vec{\ell}\,'$



$$|\vec{m}| = I\int dA = IA$$

So the magnetic moment is the product of the current and the area enclosed by the wires.



$$dA = \frac{1}{2}(r + d\ell\cos\theta)(d\ell\sin\theta) - \frac{1}{2}(d\ell\cos\theta)(d\ell\sin\theta)$$

$$= \frac{1}{2} r\, d\ell \sin\theta = \frac{1}{2}|\vec{r} \times d\vec{\ell}|$$

Suppose a small magnetic dipole is placed in an external magnetic field - we can investigate the force and torque it feels.

In general a current distribution feels a force $\vec{F} = \int d^3\vec{r}\ \vec{J}(\vec{r}) \times \vec{B}(\vec{r})$

If we consider a small size dipole & place it at the origin of our co-ordinate system, we can assume the external magnetic field varies rather little over the size of the dipole - a Taylor series is thus justified

$$B_k(\vec{r} \simeq \vec{0}) \simeq B_k(\vec{0}) + \vec{r} \cdot \underbrace{\vec{\nabla} B_k(\vec{0})}_{\text{we mean take the derivative and evaluate at the origin}} + \ldots$$

now $$F_i = \epsilon_{ijk} \int d^3r\ J_j B_k \simeq \epsilon_{ijk} B_k(\vec{0}) \underbrace{\int d^3\vec{r}\ J_j(\vec{r})}_{=0} + \epsilon_{ijk} \int d^3\vec{r}\ J_j(\vec{r})\ r_\ell \nabla_\ell B_k(\vec{0}) + \ldots$$

$$\simeq \epsilon_{ijk} \nabla_\ell B_k(\vec{0}) \underbrace{\int d^3\vec{r}\ J_j r_\ell}_{\int d^3r \frac{1}{2}(J_j r_\ell - J_\ell r_j) = \frac{1}{2}\epsilon_{\ell jm}(\vec{r} \times \vec{J})_m}$$

$$\simeq \epsilon_{ijk} \epsilon_{\ell jm} \nabla_\ell B_k(\vec{0}) \underbrace{\left[\frac{1}{2}\int d^3\vec{r}\,(\vec{r} \times \vec{J})\right]_m}_{m_m} = +\epsilon_{ijk}\,\epsilon_{\ell jm}\ m_m \nabla_\ell B_k$$

$$F_i = +(\delta_{i\ell}\delta_{km} - \delta_{im}\delta_{k\ell})\, m_m \nabla_\ell B_k = \nabla_i(\vec{m} \cdot \vec{B}) - m_i \underbrace{\vec{\nabla} \cdot \vec{B}}_{=0}$$

$$\boxed{\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})}$$ force (to lowest order) on a dipole $\vec{B}$

N.B. if the external field is uniform then $\vec{F} = \vec{0}$ as we'd expect by analogy to the electrostatic dipole.

for a conservative force, the force is the negative gradient of a potential energy

$$\vec{F} = -\vec{\nabla} U \qquad \& \text{ thus } \qquad U = -\vec{m} \cdot \vec{B}$$

the torque follows analogously from $\vec{\tau} = \int d^3\vec{r}\ \vec{r} \times (\vec{J} \times \vec{B}) = \vec{m} \times \vec{B}(\vec{0})$
where we see that there will be a torque even for a uniform field