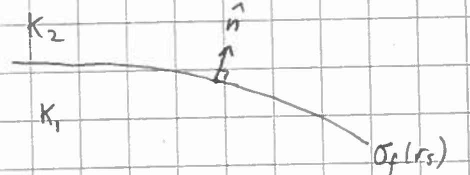
POTENTIAL THEORY

In a simple dielectric medium with dielectric constant K, the potential satisfies Poisson's equation

$$\nabla^2 \phi(\vec{r}) = -\frac{\rho_f(\vec{r})}{K\varepsilon_0} \qquad (P)$$

& matching conditions describe the behaviour of the potential at interfaces where the dielectric constant may change & where there may be free surface charge density :-

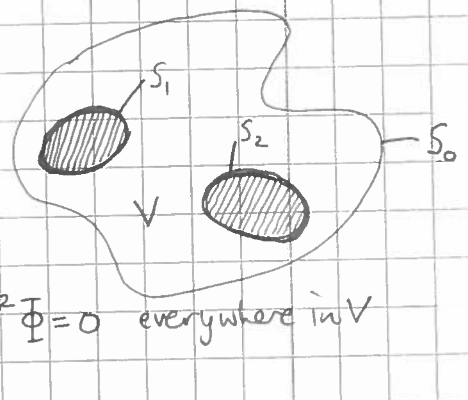


$$\phi_1(\vec{r}_S) = \phi_2(\vec{r}_S)$$

$$K_1 \frac{\partial \phi_1}{\partial n}\bigg|_S - K_2 \frac{\partial \phi_2}{\partial n}\bigg|_S = \sigma_f/\varepsilon_0 \qquad \left(\text{from } \hat{n}\cdot(\vec{D}_2 - \vec{D}_1) = \sigma_f \ \& \ \vec{D} = K\varepsilon_0(-\vec{\nabla}\phi)\right)$$

We might worry that there might be multiple solutions to (P) even when we specify appropriate boundary conditions. We can show that this isn't the case –

Consider a volume, V, which might contain a volume density of free charge, $\rho(\vec{r})$, and which might contain some conductors which have surfaces, S_i



Suppose that (P) has two distinct solutions, $\phi_A(\vec{r})$ & $\phi_B(\vec{r})$,

$$-\nabla^2\phi_A = \rho_f/\varepsilon \quad \& \quad -\nabla^2\phi_B = \rho_f/\varepsilon$$

Consider the difference $\Phi(\vec{r}) = \phi_A(\vec{r}) - \phi_B(\vec{r})$, then clearly $\nabla^2\Phi = 0$ everywhere in V

In Ex 3(c) in the homework we showed that

$$\int_V d^3\vec{r}\left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) = \int d\vec{S}\cdot\left(\phi\vec{\nabla}\psi - \psi\vec{\nabla}\phi\right)$$ & if we integrate by parts the term $\psi\nabla^2\phi$

we get $$\int_V d^3\vec{r}\left(\phi\nabla^2\psi + \vec{\nabla}\psi\cdot\vec{\nabla}\phi\right) = \int d\vec{S}\cdot\left(\phi\vec{\nabla}\psi\right)$$

so if we set $\phi = \psi = \Phi$ & use $\nabla^2\Phi = 0$ we obtain $$\int_V d^3\vec{r}\,|\vec{\nabla}\Phi|^2 = \sum_i \int_{S_i} d\vec{S}\cdot(\Phi\vec{\nabla}\Phi)$$

the RHS is zero for the following boundary conditions : (D) "Dirichlet" : specify the value of ϕ at all points on the surfaces. then $\Phi = \phi_A - \phi_B = 0$ on the surfaces

(N) "Neumann" : specify the normal component of the gradient of the potential he RHS is zero then $\int_V d^3\vec{r}\ |\vec{\nabla}\Phi|^2 = 0$.

nce this is an integral ver non-negative function it must be that $\vec{\nabla}\Phi =$
everywhere in V. Th t must be that

$$\phi_A(\vec{r}) = \phi_B(\vec{r}) +$$

and since constant shifts of the potential are not physically releva we
have that $\phi_A(\vec{r})$ is a UNIQUE solution.

Separation of variables in CARTESIAN coordinates

Potential problems featuring rectangular boundaries are most naturally solved using cartesian coordinates. Suppose we consider a rectangular region of space containing no free charge, then

$$\nabla^2\phi = 0 \quad \rightarrow \quad \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

& we might guess that a trial solution of the form $\phi(x,y,z) = X(x)\,Y(y)\,Z(z)$ might be capable of solving Laplace's eqn

$$\underbrace{\frac{X''(x)}{X(x)}} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = 0$$

↳ not a function of y, z ⇒ can only be a function of x, but a fn of x can't cancel the remaining two terms

⇒ must be a constant

$$X''/X = \alpha^2, \quad Y''/Y = \beta^2, \quad Z''/Z = \gamma^2 \quad \& \quad \alpha^2 + \beta^2 + \gamma^2 = 0$$

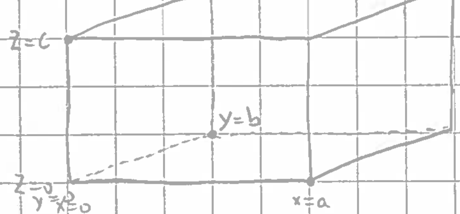
these equations can be solved easily: if $\alpha = 0$: $X_\alpha(x) = A_0 + B_0 x$

if $\alpha \neq 0$: $X_\alpha(x) = A_\alpha e^{\alpha x} + B_\alpha e^{-\alpha x}$

(N.B if $\alpha^2 < 0$ & α is imaginary this can be expressed as a sum of sin, cos)

& similarly for Y, Z.

This approach is best illustrated by a concrete example: consider a rectangular box



where all the walls are held at a potential of zero except the bottom wall (at $z=0$) which has a potential $V(x,y)$.

for the vertical walls at $x=0$ & $x=a$: $\phi(0,y,z) = X_\alpha(0)\,Y_\beta(y)\,Z_\gamma(z) = 0$ & $\phi(a,y,z) = X_\alpha(a)\,Y_\beta(y)\,Z_\gamma(z) =$

$\left.\begin{array}{l} X_\alpha(0) = 0 \\ X_\alpha(a) = 0 \end{array}\right\}$ ⇒ we need the periodic solutions from $\alpha^2 < 0$ $\quad X_\alpha(x) = \tilde{A}_\alpha \sin\alpha x + \tilde{B}_\alpha \cos\alpha x$

$X_\alpha(0) = 0 \Rightarrow \tilde{B}_\alpha = 0$

$X_\alpha(a) = 0 \Rightarrow \sin\alpha a = 0 \Rightarrow \alpha = m\pi/a$

solutions allowed by the boundary conditions:

$$X(x) \propto \sin\frac{m\pi x}{a}$$

milarly for the other vertical walls so that $Y(y) \propto \sin\frac{n\pi y}{b}$

Since $\alpha^2+\beta^2+\gamma^2=0$ we're left with

$$Z(z) = E_{mn} e^{\gamma_{mn} z} + F_{mn} e^{-\gamma_{mn} z}$$

$\rightarrow \gamma_{mn}^2 = \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}$

$$0 = Z(c) = E_{mn} e^{\gamma c} + F_{mn} e^{-\gamma c} \Rightarrow E_{mn} = -F_{mn} e^{-2\gamma c}$$

$$\& \; Z(z) = F_{mn}\left(-e^{-2\gamma c} e^{\gamma z} + e^{-\gamma z}\right)$$

$$= -F_{mn} e^{-\gamma c}\left(e^{\gamma z} e^{-\gamma c} - e^{-\gamma z} e^{\gamma c}\right)$$

$$= -F_{mn} e^{-\gamma c}\, 2\sinh \gamma(z-c)$$

& making a convenient definition of the undetermined constant F_{mn} we can write

$$Z(z) \propto \frac{\sinh[\gamma_{mn}(c-z)]}{\sinh[\gamma_{mn} c]}$$

such that the general solution of $\nabla^2\phi = 0$ satisfying the boundary conditions on all walls EXCEPT the bottom one is

$$\phi(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_{mn} \sin\left(\frac{m\pi x}{a}\right)\cdot \sin\left(\frac{n\pi y}{b}\right)\cdot \frac{\sinh[\gamma_{mn}(c-z)]}{\sinh[\gamma_{mn} c]}$$

applying the b.c on the bottom wall then

$$V(x,y) = \phi(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

which is clearly the Fourier series for $V(x,y)$

Considering $$\int_0^a dx \int_0^b dy \sin\frac{\bar{m}\pi x}{a} \sin\frac{\bar{n}\pi y}{b} V(x,y) = \sum_{m,n} V_{mn}\left[\int_0^a dx \sin\frac{\bar{m}\pi x}{a}\sin\frac{m\pi x}{a}\right]\left[\int_0^b dy \sin\frac{\bar{n}\pi y}{b}\sin\frac{n\pi y}{b}\right]$$

$$= \sum_{m,n} V_{mn}\left(\frac{a}{2}\delta_{\bar{m}m}\right)\left(\frac{b}{2}\delta_{\bar{n}n}\right)$$

$$\Rightarrow V_{mn} = \frac{4}{ab}\int_0^a dx \int_0^b dy\, V(x,y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

Separation for systems with AZIMUTHAL symmetry

by which we mean systems which when expressed in spherical polar coordinates have dependence only on r and θ but not ϕ.

thus $\frac{\partial}{\partial\phi}\phi(r,\theta,\phi) = 0$ & $\nabla^2\phi(r,\theta) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\phi + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\phi$

using the shorthand $x \equiv \cos\theta$ we have $\frac{\partial}{\partial\theta} = \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} = -\sin\theta\frac{\partial}{\partial x}$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) = -\frac{\partial}{\partial x}\left(\sin\theta\frac{\partial}{\partial\theta}\right) = -\frac{\partial}{\partial x}\left(-\sin^2\theta\frac{\partial}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial}{\partial x}\right]$$

$$\nabla^2\phi = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial\phi}{\partial x}\right]\right]$$

& we can separate $\phi(r,x) = R(r)F(x)$ to solve $\nabla^2\phi = 0$

$$0 = \underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{+K} + \underbrace{\frac{1}{F}\frac{d}{dx}\left((1-x^2)\frac{dF}{dx}\right)}_{-K}$$

& if we (arbitrarily) choose to write $K = \nu(\nu+1)$ we have

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \nu(\nu+1)R \qquad \& \qquad (x^2-1)\frac{d^2F}{dx^2} + 2x\frac{dF}{dx} - \nu(\nu+1)F = 0$$

which is solved by

$$R_\nu(r) = A_\nu r^\nu + B_\nu\frac{1}{r^{\nu+1}}$$

which is Legendre's equation which has polynomial solutions for integer ν

without going into details, insisting that the solution be valid everywhere (including the z-axis) removes the possibility of "Legendre fns of the second kind" ($Q_\ell(x)$) and insists that ν take integer values

$$\Rightarrow \quad \phi(r,\theta) = \sum_{\ell=0}^{\infty}\left[A_\ell r^\ell + B_\ell\frac{1}{r^{\ell+1}}\right]P_\ell(\cos\theta)$$

9. a azimuthally symmetric charge distribution on the surface of a sphere of radius R

1. find most general $\phi(r,\theta)$

- for $r<R$ ~ must not be singular as $r\to 0$
- for $r>R$ ~ must die off as $r\to\infty$

2. match these solutions at $r=R$ recalling that there's a charge here

$$\phi(r,\theta) = \begin{cases} \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) & r<R \\ \sum_{\ell=0}^{\infty} B_\ell \frac{1}{r^{\ell+1}} P_\ell(\cos\theta) & r>R \end{cases}$$

2. the potential must be continuous @ $r=R$ $\Rightarrow A_\ell R^\ell = B_\ell \, {}^{1}/_{R^{\ell+1}} \Rightarrow \underline{B_\ell = A_\ell R^{2\ell+1}}$

the radial derivative of the potential is discontinuous since $\hat{n}\cdot(\vec{E}_2-\vec{E}_1) = \sigma_f/\varepsilon_0$
with region 1 inside the sphere
& region 2 outside the sphere $\Rightarrow \hat{r}\cdot\left(-\vec{\nabla}\phi_{out} + \vec{\nabla}\phi_{in}\right) \; \sigma_f/\varepsilon_0$

$$\left.\frac{\partial\phi_{in}}{\partial r}\right|_R - \left.\frac{\partial\phi_{out}}{\partial r}\right|_R = \sigma_f/\varepsilon_0$$

$$\Rightarrow \sum_\ell \ell A_\ell R^{\ell-1} P_\ell(\cos\theta) - \sum_\ell -(\ell+1) A_\ell R^{2\ell+1} \frac{1}{R^{\ell+2}} P_\ell(\cos\theta) = \frac{\sigma(\theta)}{\varepsilon_0}$$

$$\varepsilon_0 \frac{1}{R} \sum_\ell (2\ell+1) A_\ell R^\ell P_\ell(\cos\theta) = \sigma(\theta)$$

& since $\int_{-1}^{1} dx \, P_\ell(x) P_{\ell'}(x) = \frac{2}{2\ell+1}\delta_{\ell\ell'}$ we have $A_\ell = \frac{R^{1-\ell}}{2\varepsilon_0} \cdot \int_0^\pi \sin\theta \, d\theta \, \sigma(\theta) P_\ell(\cos\theta)$

e.g. a uniform loop of charge lying in the x-y plane at $z=0$: $\sigma(\theta) = \frac{Q}{2\pi R^2}\,\delta(\cos\theta)$

$$\Rightarrow \quad A_\ell = \frac{R^{1-\ell}}{2\varepsilon_0}\cdot\frac{Q}{2\pi R^2}\int_{-1}^{+1} d\cos\theta\;\delta(\cos\theta)\;P_\ell(\cos\theta) \;=\; \frac{Q}{4\pi\varepsilon_0}\,\frac{1}{R^{\ell+1}}\,P_\ell(0)$$

$$B_\ell = \frac{Q}{4\pi\varepsilon_0} R^\ell P_\ell(0)$$

$$\Rightarrow \quad \phi(r,\theta) = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0}\displaystyle\sum_{\ell=0}^{\infty}\frac{r^\ell}{R^{\ell+1}}\,P_\ell(0)\,P_\ell(\cos\theta) & r\le R \\[2ex] \dfrac{Q}{4\pi\varepsilon_0}\displaystyle\sum_{\ell=0}^{\infty}\frac{R^\ell}{r^{\ell+1}}\,P_\ell(0)\,P_\ell(\cos\theta) & r\ge R \end{cases}$$

check: on the symmetry axis $\underset{r\to z}{(\theta=0)} \longrightarrow \phi(z) = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0}\dfrac{1}{R}\displaystyle\sum_{\ell=0}^{\infty}\left(\frac{z}{R}\right)^\ell P_\ell(0) & z\le R \\[2ex] \dfrac{Q}{4\pi\varepsilon_0}\dfrac{1}{z}\displaystyle\sum_{\ell=0}^{\infty}\left(\frac{R}{z}\right)^\ell P_\ell(0) & z\ge R \end{cases}$

Which you can check are the series expansions of $\phi(z) = \frac{Q}{4\pi\varepsilon_0}\,\frac{1}{\sqrt{z^2+R^2}}$

which we can obtain by more elementary methods

(e.g. $\phi(\vec r) = \frac{1}{4\pi\varepsilon_0}\int dS'\,\frac{\sigma(\vec r_s')}{|\vec r - \vec r_s'|}$)

Separation for systems with SPHERICAL symmetry

Since $\nabla^2 \phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\phi = 0$

we can separate using $\phi(r,\theta,\phi) = R(r)\, Y(\theta,\phi)$ & if we use $\ell(\ell+1)$ as the separation constant we get that the functions $Y_{\ell m}(\theta,\phi)$ are the spherical harmonics we discussed earlier and

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = \ell(\ell+1) R \qquad \text{so as before} \qquad R_\ell(r) = A_\ell r^\ell + B_\ell \frac{1}{r^{\ell+1}}.$$

$$\& \quad \phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(A_{\ell m} r^\ell + B_{\ell m}\frac{1}{r^{\ell+1}}\right) Y_{\ell m}(\theta,\phi)$$

which we note is exactly what we expect for the "interior" & "exterior" multipole expansions we saw earlier

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Separation for systems with CYLINDRICAL symmetry

$$\nabla^2 \varphi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \varphi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \varphi}{\partial \phi^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \qquad \varphi(\rho,\phi,z) = R(\rho)\, G(\phi)\, Z(z)$$

$$\frac{1}{\rho R}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \underbrace{\frac{1}{\rho^2 G}\frac{d^2 G}{d\phi^2}}_{-\frac{1}{\rho^2}\alpha^2} + \underbrace{\frac{1}{Z}\frac{d^2 Z}{dz^2}}_{k^2} = 0$$

$$Z'' = k^2 Z \quad ; \quad G'' = -\alpha^2 G \quad ; \quad \rho\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + (k^2\rho^2 - \alpha^2) R = 0$$

which is the Bessel differential equation

when $k^2 > 0$, the general solution is $R_{\alpha k}(\rho) = A_{\alpha k} J_\alpha(k\rho) + B_{\alpha k} N_\alpha(k\rho)$

if $k=0$ we have the special case $\rho\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) = \alpha^2 R \xrightarrow{\alpha=0} R = A + B\ln\rho$

$\xrightarrow{\alpha\neq 0} A_\alpha \rho^\alpha + B_\alpha \frac{1}{\rho^\alpha}$

→ Continuity in $\phi \Rightarrow \alpha = n$ = non-negative integer

9 the potential inside a cylinder of radius R & height L with the potential being zero on the bottom face & the curved side and having form $V(\rho, \phi)$ on the top face.

+ appropriate solutions of $G'' = -\alpha^2 G$ are $\underline{G(\phi) = A \sin m\phi + B \cos m\phi}$ (where m = integer ensures single valued)

the appropriate solutions of $Z'' = k^2 Z$ are $\underline{Z(z) = C \sinh kz + D \cosh kz}$

but since $\varphi(\rho, \phi, z=0) = 0 \Rightarrow \underline{Z(z) \to \sinh kz}$

in order that the potential remain finite at $\rho = 0$, we must reject the $N_\alpha(k\rho)$ solutions

& $R(\rho) \to J_m(k\rho)$

on applying the boundary condition $\varphi(\rho = R, \phi, 0 \le z \le L) = 0$

we have $0 = J_m(kR)$

so $kR = a_n^{(m)}$ ← position of the zeroes of $J(x)$

$$\underline{k_{mn} = a_n^{(m)}/R}$$

and thus the most general solution having applied the bottom & side boundary conditions is

$$\varphi(\rho, \phi, z) = \sum_{m=0}^{\infty} \sum_{n=}^{\infty} J\,(k_{mn}\rho) \sinh(k_{mn} z) \left[A_{mn} \sin m\phi + B_{mn} \cos m\phi \right]$$

the remaining boundary condition is

$$V(\rho, \phi) = \sum_{m,n} J_m(k_{mn}\rho) \sinh(k_{mn} L) \left[A_{mn} \sin m\phi + B_{mn} \cos \phi \right]$$

which we should solve for A_{mn}, B_{mn}.

we need the orthogonality relations

$$\int_0^{2\pi} d\phi \, \sin \bar{m}\phi \sin m\phi = \pi \delta_{m,\bar{m}}$$

$$\int_0^{2\pi} d\phi \, \sin \bar{m}\phi \cos m\phi = 0$$

$$\int_0^{2\pi} d\phi \, \cos \bar{m}\phi \cos m\phi = \pi \delta_{m,\bar{m}} \quad \text{for } m \ne 0$$

and $\int^1 dx \, x \, J_0(a_n^{(0)} x) \, J_0(a_m^{(0)} x) \;\; \delta \;\; (J'(a^0_n))^2$

then

$$\int_0^{2\pi} d\phi \int_0^R \rho\, d\rho \; V(\rho,\phi)\, J_{\bar m}(k_{\bar m \bar n}\rho)\, \sin \bar m\phi$$

$$= \sum_{m,n} \sinh(k_{mn}L) A_{mn} \underbrace{\int_0^{2\pi} d\phi\, \sin \bar m\phi \sin m\phi}_{\pi\delta_{m\bar m}} \; \underbrace{\int_0^R \rho\, d\rho\; J_{\bar m}(k_{\bar m\bar n}\rho) J_m(k_{mn}\rho)}_{x=\rho/R}$$

$$R^2 \int_0^1 dx\; x\; J_{\bar m}\left(a_{\bar n}^{(\bar m)} x\right) J_m\left(a_n^{(m)} x\right)$$

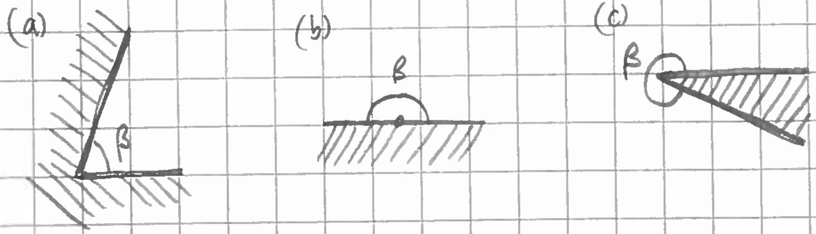
$$= R^2 \int_0^1 dx\; x\; J_m\left(a_{\bar n}^{(m)} x\right) J_m\left(a_n^{(m)} x\right) = R^2 \frac{1}{2}\left(J_m'(a_n^{(m)})\right)^2 \times \delta_{n\bar n}$$

$$\Rightarrow A_{mn} = \frac{2}{\pi R^2} \cdot \frac{1}{\sinh(k_{mn}L)} \cdot \frac{1}{\left(J_m'(k_{mn}R)\right)^2} \cdot \int_0^{2\pi} d\phi \int_0^R \rho\, d\rho\; V(\rho,\phi)\, J_m(k_{mn}\rho) \sin m\phi$$

& similarly for B_{mn}.

a "Fourier-Bessel" series.

Electric field near sharp corners or edges



are examples of grounded conductors that continue infinitely into & out of the page

Using cylindrical coordinates we have $Z(z) = 1$ & $k^2 = 0$

$$G(\phi) = A\cos\alpha\phi + B\sin\alpha\phi = C_\alpha \sin(\alpha\phi + \delta_\alpha)$$

[the $A + B\phi$ possibility can be rejected because $\varphi(\rho=0,\phi)=0$]

$$\& \quad \rho\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) - \alpha^2 R = 0 \Rightarrow R(\rho) = A_\alpha \rho^\alpha + B_\alpha \frac{1}{\rho^\alpha}$$

but since we need the solution to be valid as $\rho \to 0$ we can reject the $\frac{1}{\rho^\alpha}$ possibility

and conclude $$\varphi(\rho,\phi) = \sum_\alpha C_\alpha \rho^\alpha \sin(\alpha\phi + \delta_\alpha)$$

at two angles $\phi = 0, \beta$ $\forall \rho$ the potential is zero $\quad \varphi(\rho,0) = 0 = \sum_\alpha C_\alpha \rho^\alpha \sin\delta_\alpha \Rightarrow \delta_\alpha = 0$

$$\varphi(\rho,\beta) = 0 = \sum_\alpha C_\alpha \rho^\alpha \sin\alpha\beta \Rightarrow \alpha\beta = m\pi$$
$$\alpha = m\pi/\beta$$

$$\varphi(\rho,\phi) = \sum_{m=1}^{\infty} C_m \rho^{m\pi/\beta} \sin\left(m\frac{\pi}{\beta}\phi\right)$$

for small values of ρ, the $m=1$ term will dominate $\varphi(\rho,\phi) \to C_1 \rho^{\pi/\beta} \sin(\pi\phi/\beta)$

$$\& \quad \vec{E} = -\vec{\nabla}\varphi = -\left(\hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi}\right)\left(C_1 \rho^{\pi/\beta}\sin(\pi\phi/\beta)\right)$$

$$= \left(-C_1 \frac{\pi}{\beta}\rho^{\pi/\beta - 1} \sin \pi\phi/\beta\right)\hat{\rho} + \left(-C_1\frac{\pi}{\beta}\rho^{\pi/\beta - 1}\cos\pi\phi/\beta\right)\hat{\phi}$$

$$\underline{\vec{E} = -\frac{C_1\pi}{\beta}\rho^{\pi/\beta - 1}\left\{\hat{\rho}\sin\pi\phi/\beta + \hat{\phi}\cos\pi\phi/\beta\right\}}$$

e.g. $\beta = \pi \Rightarrow$ conducting plane, $\vec{E} = -C_1(\hat{\rho}\sin\phi + \hat{\phi}\cos\phi) = -C_1\hat{y}$ ✓

if $\beta < \pi$, e.g. a "notch" as in (a) $\quad \vec{E}(\rho \to 0) \to 0$

if $\beta > \pi$, e.g. an "edge" as in (c) $\quad \vec{E}(\rho\to 0) \to \rho^{-(1-\pi/\beta)} \to \infty$

POISSON'S EQUATION – "what if there are charges"

Poisson's equation for the potential in vacuum is $\nabla^2 \varphi(\vec{r}) = -\rho(\vec{r})/\varepsilon_0$.

The complications with solving this equation come when we introduce charge into a volume containing conductors or dielectric material. In this case charge is rearranged in the conductors and the dielectrics and $\rho(\vec{r})$ becomes complicated.

Let's begin with the simple case of a single point charge q at position $\vec{r}_0$ which lies in a volume V where the potential on the boundary of V is known. Then the solution of Poisson's equation must be of the form

$$\varphi(\vec{r}) = \frac{q}{4\pi\varepsilon_0 |\vec{r}-\vec{r}_0|} + \varphi_0(\vec{r}) \qquad \text{where } \nabla^2\varphi_0 = 0 \qquad (A)$$

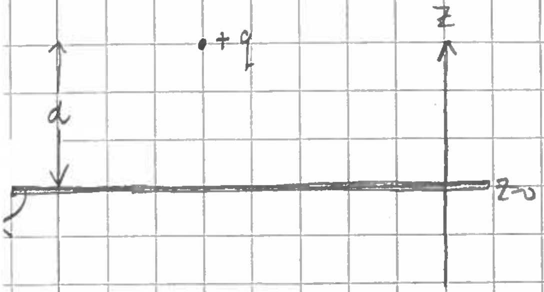
by construction $\quad \nabla^2 \varphi = -q\,\delta(\vec{r}-\vec{r}_0)/\varepsilon_0$

The role of $\varphi_0(\vec{r})$ is to ensure that the potential on the boundary of V is what the boundary conditions imply.

e.g. suppose the boundary of V is a grounded conductor, then $\varphi = 0$ on it – in this case $\varphi_0(\vec{r})$ corresponds to the potential caused by the charge distribution induced on the surface of the conductor.

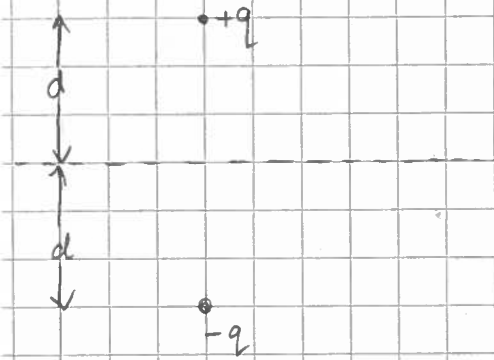
We'll see later, when we consider "Green's functions" that an arbitrary $\rho(\vec{r})$ can be built up from point charges, so let's proceed with this simple case.

The method of images : point charge near a conducting plane



Find the potential for all $z>0$.

We propose that the potential for $z>0$ is the same as it would be for the following system



in which there is no conductor, but rather an "image" charge

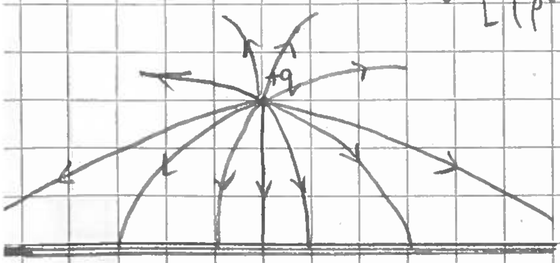
Clearly $z=0$ is still an equipotential with $\varphi=0$, and for general points $z\geq 0$:

$$\varphi(\rho, z\geq 0) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{\rho^2+(z-d)^2}} - \frac{1}{\sqrt{\rho^2+(z+d)^2}}\right] \quad (B)$$

and notice that the first term is the particular solution of Poisson's eqn with a charge at $z=d$, which for $z\geq 0$, the second term is a solution of Laplace's eqn, i.e. this solution is of the form (A) & by construction satisfies the boundary condition $\varphi(\rho, z=0)=0$.

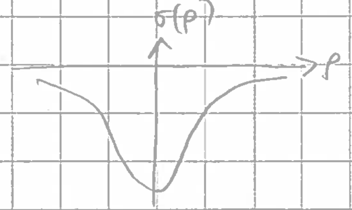
Thus, remarkably, we are done - we've solved the original problem, the electric field can easily be found:

$$\vec{E}(\rho, z\geq 0) = \frac{q}{4\pi\epsilon_0}\left[\frac{\rho\hat{\rho} + (z-d)\hat{z}}{\left(\rho^2+(z-d)^2\right)^{3/2}} - \frac{\rho\hat{\rho} + (z+d)\hat{z}}{\left(\rho^2+(z+d)^2\right)^{3/2}}\right]$$



the "physical" source of the second term in (B) is the surface charge induced on the conductor's surface, which we can find since

$$\vec{E}(\rho, z=0) = \frac{\sigma}{\varepsilon_0}\hat{z}$$



$$\sigma = \varepsilon_0 \hat{z}\cdot\vec{E}(\rho, z=0) = \frac{q}{4\pi}\frac{-2d}{(\rho^2+d^2)^{3/2}} = \frac{-q\,d}{2\pi(\rho^2+d^2)^{3/2}}$$

& the total charge drawn up from ground is $q' = 2\pi\int_0^\infty \rho\,d\rho\,\sigma(\rho) = -q\,d\int_0^\infty d\rho\,\frac{\rho}{(\rho^2+d^2)^{3/2}} = -q$

$\Rightarrow$ all field lines terminate on the conducting plane.

---

we might wonder how our $\varphi_0 = \frac{-q}{4\pi\varepsilon_0}\frac{1}{\sqrt{\rho^2+(z+d)^2}}$ which satisfies $\nabla^2\varphi = 0$ for $z \geq 0$

is related to the variable separated solutions with cylindrical symmetry we considered previously. In general we expect

$$\varphi_0(\rho, z\geq 0) = \int_0^\infty dk\,A(k)\,J_0(k\rho)\,e^{-kz}$$

where we integrate over possible value of k since we have no boundary condition in ρ & hence must allow all possibilities. The e^{-kz} solution choice is dictated by the need for $\varphi_0(\rho, z\to\infty)\to 0$.

In HW1 we proved the identity $\frac{1}{\sqrt{\rho^2+z^2}} = \int_0^\infty dk\,J_0(k\rho)\,e^{-k|z|}$

thus $\varphi(\rho, z\geq 0) = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{\rho^2+(z-d)^2}} + \int_0^\infty dk\,A(k)\,J_0(k\rho)\,e^{-kz}$

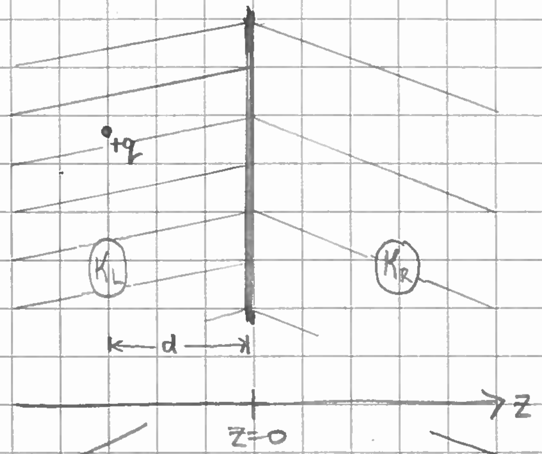
$$= \frac{q}{4\pi\varepsilon_0}\int_0^\infty dk\,J_0(k\rho)\,e^{-k|z-d|} + \int_0^\infty dk\,A(k)\,J_0(k\rho)\,e^{-kz}$$

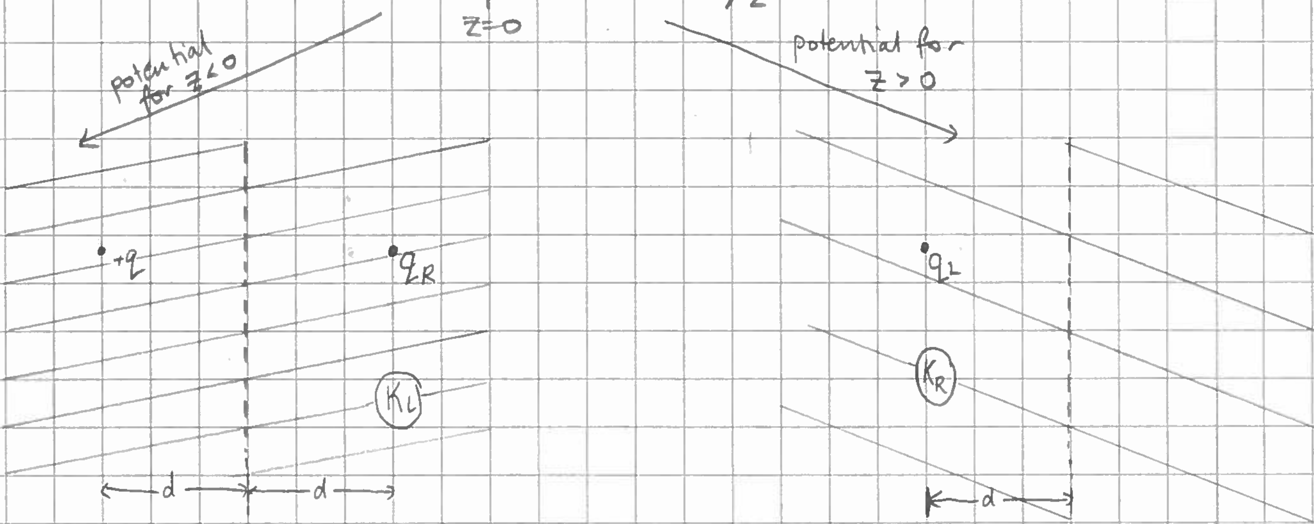
then $\varphi(\rho, z=0) = 0$ if $A(k) = -\frac{q}{4\pi\varepsilon_0}e^{-kd}$

then $\varphi(\rho, z\geq 0) = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{\rho^2+(z-d)^2}} - \frac{q}{4\pi\varepsilon_0}\int_0^\infty dk\,J_0(k\rho)\,e^{-k(z+d)} = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{\rho^2+(z-d)^2}} - \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{\rho^2+(z+d)^2}}$ ✓

The method of images: point charge near a dielectric boundary

$\varepsilon_L = K_L \varepsilon_0$
$\varepsilon_R = K_R \varepsilon_0$





$$\varphi(\rho, z\le 0) = \frac{1}{4\pi\varepsilon_L}\left[\frac{q}{\sqrt{\rho^2+(z+d)^2}} + \frac{q_R}{\sqrt{\rho^2+(z-d)^2}}\right] \qquad \varphi(\rho, z\ge 0) = \frac{1}{4\pi\varepsilon_R}\frac{q_L}{\sqrt{\rho^2+(z+d)^2}}$$

the potential is continuous at the interface $\Rightarrow$ $\dfrac{q+q_R}{\varepsilon_L} = \dfrac{q_L}{\varepsilon_R}$ (A)

the perpendicular $\vec{D}$ field is continuous at the interface since there's no free charge there

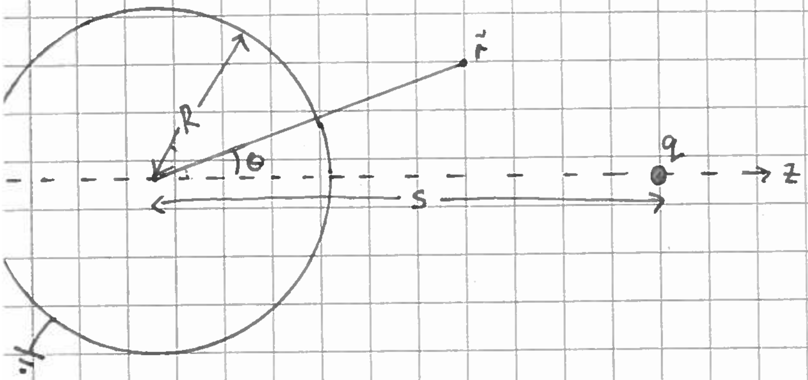
$$\Rightarrow K_L \frac{\partial\varphi(z<0)}{\partial z} = K_R \frac{\partial \varphi(z>0)}{\partial z} \Longrightarrow q - q_R = q_L \quad (B)$$

solving (A),(B) $\longrightarrow$ $q_R = \dfrac{K_L - K_R}{K_L+K_R}\, q$, $q_L = \dfrac{2K_R}{K_L+K_R}\, q$

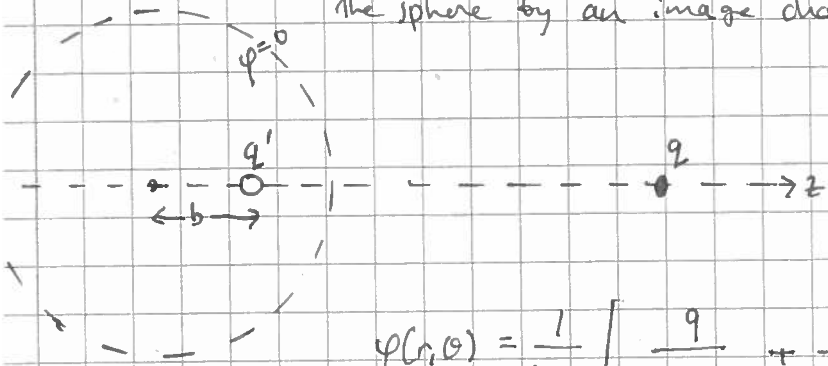
HW: find the polarisation charge on the interface.

The method of images: point charge outside a conducting sphere

e.g. a grounded conducting sphere



potential outside the sphere the same if we replace the sphere by an image charge



$$\varphi(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\vec r-\vec s|} + \frac{q'}{|\vec r-\vec b|}\right] \qquad \begin{aligned}\vec s &= s\,\hat z\\ \vec b &= b\,\hat z\end{aligned}$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{s^2+r^2-2sr\cos\theta}} + \frac{q'}{\sqrt{b^2+r^2-2br\cos\theta}}\right]$$

to determine q', b we should apply the condition $\varphi(R,\theta)=0$. There are some simple tricks to achieve this, but let's use a more general method utilizing the expansion $\frac{1}{\sqrt{r'^2+r^2-2rr'\cos\theta}} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\theta)$ for $r>r'$

$$0 = \varphi(R,\theta) = \frac{q}{s}\sum_{l=0}^{\infty}\left(\frac{R}{s}\right)^l P_l(\cos\theta) + \frac{q'}{R}\sum_{l=0}^{\infty}\left(\frac{b}{R}\right)^l P_l(\cos\theta)$$

$$= \sum_{l=0}^{\infty} P_l(\cos\theta)\left[\frac{q}{s}\left(\frac{R}{s}\right)^l + \frac{q'}{R}\left(\frac{b}{R}\right)^l\right]$$

& since this must hold for all θ, it must be that each coefficient of $P_l(\cos\theta)$ vanishes

$l=0$: $\frac{q}{s} = -q'/R \Rightarrow \boxed{q' = -q\,R/s}$

$l=1$: $qR/s^2 = -q'\,b/R^2 = q\,R/s\;b/R^2 = q\,b/sR \Rightarrow \boxed{b = R^2/s}$

$\Rightarrow$ for general l: $\frac{q}{s}\left(\frac{R}{s}\right)^l - \frac{qR}{sR}\left(\frac{R}{s}\right)^l = 0$ ✓

$$\varphi(r \geq R, \theta) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{r}-\vec{s}|} - \frac{R/s}{|\vec{r} - \frac{R^2}{s^2}\vec{s}|}\right]$$

note that $|q'| < |q|$ implying that the charge drawn up from ground does not completely neutralize the system. This is because not all field lines from q terminate on the sphere, some escape to infinity.

Green's Functions

Green's functions are a tool that allows us to write a general solution of $\nabla^2\varphi(\vec{r}) = -\rho(\vec{r})/\varepsilon_0$ building in the Dirichlet or Neumann boundary conditions that apply on the boundary of the volume considered.

Define the function $G(\vec{r},\vec{r}\,')$ which satisfies $\nabla^2 G(\vec{r},\vec{r}\,') = -\frac{1}{\varepsilon_0}\delta(\vec{r}-\vec{r}\,')$ [G]

and which satisfies the appropriate boundary conditions.

In the first homework we derived "Green's second identity"

$$\int_V d^3\vec{r}\,(f\nabla^2 g - g\nabla^2 f) = \int_S d\vec{S}\cdot(f\vec{\nabla}g - g\vec{\nabla}f)$$

then if we use primed variables as the dummy variables & choose $f(\vec{r}\,') = \varphi(\vec{r}\,')$
& $g(\vec{r}\,') = G(\vec{r}\,',\vec{r})$

NB arguments reversed

we obtain

$$\int_V d^3\vec{r}\,'\,\varphi(\vec{r}\,')\underbrace{\nabla^2_{\vec{r}\,'}G(\vec{r}\,',\vec{r})}_{-\frac{1}{\varepsilon_0}\delta(\vec{r}-\vec{r}\,')} - \int_V d^3\vec{r}\,'\,G(\vec{r}\,',\vec{r})\underbrace{\nabla^2_{\vec{r}\,'}\varphi(\vec{r}\,')}_{-\rho(\vec{r}\,')/\varepsilon_0}$$

$$= \int_S dS'\,\varphi(\vec{r}\,')\,\hat{n}'\cdot\vec{\nabla}G(\vec{r}\,',\vec{r}) - \int_S dS'\,G(\vec{r}\,',\vec{r})\,\hat{n}'\cdot\vec{\nabla}\varphi(\vec{r}\,')$$

$$\Rightarrow \varphi(\vec{r}) = \int_V d^3\vec{r}\,'\,G(\vec{r}\,',\vec{r})\rho(\vec{r}\,') - \varepsilon_0\int_S dS'\,\varphi(\vec{r}\,')\frac{\partial}{\partial n'}G(\vec{r}\,',\vec{r}) + \varepsilon_0\int_S dS'\,\frac{\partial\varphi(\vec{r}\,')}{\partial n'}\cdot G(\vec{r}\,',\vec{r})$$

In the case of DIRICHLET boundary conditions on G we have $G_D(\vec{r}_S,\vec{r}\,') = 0$ [D]
and the last equation becomes

$$\varphi(\vec{r}) = \int_V d^3\vec{r}\,'\,G_D(\vec{r}\,',\vec{r})\rho(\vec{r}\,') - \varepsilon_0\int_S dS'\,\varphi(\vec{r}\,'_S)\frac{\partial G_D}{\partial n'}(\vec{r}\,',\vec{r})$$

and it follows that if we know $G_D(\vec{r}\,',\vec{r})$, the charge distribution $\rho(\vec{r})$ and the potential on the boundary surface, $\varphi(\vec{r}_S)$, we can find $\varphi(\vec{r})$ at all points in the volume, V, bounded by S.

(NEUMANN b.c.'s can also be implemented similarly)

Equations [G] & [D] provide a simple physical interpretation: $G_D(\vec{r},\vec{r}\,')$ is the electrostatic potential at any point $\vec{r}$ in a volume V bounded by grounded conducting walls (or infinity) due to a unit strength point charge placed at $\vec{r}\,'$ which also lies in V.

[e.g. suppose the volume is all of space, with the surface being at infinity –
then we know the potential everywhere from a unit point charge at $\vec{r}\,'$:

$$G_0(\vec{r},\vec{r}\,') = \frac{1}{4\pi\varepsilon_0}\frac{1}{|\vec{r}-\vec{r}\,'|}$$, the "free-space" Green's function]

using the symmetry $G_D(\vec{r},\vec{r}\,') = G_D(\vec{r}\,',\vec{r})$ we can rewrite our expression for the potential as

$$\boxed{\varphi(\vec{r}) = \int_V d^3\vec{r}\,'\, G_D(\vec{r},\vec{r}\,')\rho(\vec{r}\,') - \varepsilon_0\int_S dS'\, \varphi_S(\vec{r}\,')\,\frac{\partial G_D(\vec{r},\vec{r}\,')}{\partial n'}}$$

we've already obtained the Green's function for an interesting case:

charge distributed outside a conducting sphere

we found that the potential caused by a charge q at a distance s from a grounded conducting sphere was

$$\varphi(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\vec{r}-\vec{s}|} - \frac{R/s}{|\vec{r}-\frac{R^2}{s^2}\vec{s}|}\right]$$

& thus $G_D(\vec{r},\vec{r}\,') = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{|\vec{r}-\vec{r}\,'|} - \frac{R/r'}{|\vec{r}-\frac{R^2}{r'^2}\vec{r}\,'|}\right]$ $r, r' > R$

which in spherical polar coordinates where
$\vec{r} = r\sin\theta\cos\phi\,\hat{x} + r\sin\theta\sin\phi\,\hat{y} + r\cos\theta\,\hat{z}$
$\vec{r}\,' = r'\sin\theta'\cos\phi'\,\hat{x} + r'\sin\theta'\sin\phi'\,\hat{y} + r'\cos\theta'\,\hat{z}$

is

$$G_D(\vec{r},\vec{r}\,') = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{(r^2+r'^2-2rr'\cos\Theta)^{1/2}} - \frac{1}{\left(\frac{r^2r'^2}{R^2}+R^2-2rr'\cos\Theta\right)^{1/2}}\right]$$

where $\cos\Theta = \frac{\vec{r}\cdot\vec{r}\,'}{rr'} = \sin\theta\sin\theta'\cos\phi\cos\phi' + \sin\theta\sin\theta'\sin\phi\sin\phi' + \cos\theta\cos\theta' = \sin\theta\sin\theta'\cos(\phi-\phi') + \cos\theta\cos\theta'$

$$\frac{\partial G_D(\vec{r},\vec{r}\,')}{\partial n'} = \hat{n}\cdot\vec{\nabla}_{r'}G_D(\vec{r},\vec{r}\,') = (-\hat{r})\cdot\vec{\nabla}_{r'}G_D(\vec{r},\vec{r}\,') = -\frac{\partial}{\partial r'}G_D(\vec{r},\vec{r}\,')$$

& we need to evaluate this on the surface of the sphere, $r'=R$

$$\left.\frac{\partial G_D}{\partial n'}\right|_{r'=R} = \frac{-1}{4\pi\epsilon_0}\left[\frac{-\frac{1}{2}(2R-2r\cos\Theta)}{(r^2+R^2-2rR\cos\Theta)^{3/2}} - \frac{-\frac{1}{2}\left(2\frac{r^2}{R^2}R-2r\cos\Theta\right)}{(r^2+R^2-2rR\cos\Theta)^{3/2}}\right]$$

$$= \frac{-1}{4\pi\epsilon_0}\frac{-R+r^2/R}{(r^2+R^2-2rR\cos\Theta)^{3/2}} = -\frac{1}{4\pi\epsilon_0}\frac{1}{R}\frac{(r^2-R^2)}{(r^2+R^2-2rR\cos\Theta)^{3/2}}$$

an example usage might be to find the potential in a charge-free region of space when the potential is known on the surface of a sphere:

$$\varphi(\vec{r}) = +\frac{1}{4\pi R}\int dS'\,\varphi_S(\vec{r}\,')\frac{r^2-R^2}{(r^2+R^2-2rR\cos\Theta)^{3/2}}$$

$$= \frac{+R}{4\pi}(r^2-R^2)\int d\Omega'\,\varphi(R,\theta',\phi')\frac{1}{(r^2+R^2-2rR\cos\Theta)^{3/2}}$$

we've already obtained the Green's function for an interesting case:

charge distributed outside a conducting sphere

we found that the potential caused by a charge q at a distance s from a grounded conducting sphere was

$$\varphi(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{r}-\vec{s}|} - \frac{R/s}{\left|\vec{r}-\frac{R^2}{s^2}\vec{s}\right|}\right]$$

& thus $$G_D(\vec{r},\vec{r}\,') = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{r}-\vec{r}\,'|} - \frac{R/r'}{\left|\vec{r}-\frac{R^2}{r'^2}\vec{r}\,'\right|}\right] \qquad r, r' > R$$

which in spherical polar coordinates where
$$\vec{r} = r\sin\theta\cos\phi\,\hat{x} + r\sin\theta\sin\phi\,\hat{y} + r\cos\theta\,\hat{z}$$
$$\vec{r}\,' = r'\sin\theta'\cos\phi'\,\hat{x} + r'\sin\theta'\sin\phi'\,\hat{y} + r'\cos\theta'\,\hat{z}$$

is

$$G_D(\vec{r},\vec{r}\,') = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{\left(r^2+r'^2-2rr'\cos\Theta\right)^{1/2}} - \frac{1}{\left(\frac{r^2r'^2}{R^2}+R^2-2rr'\cos\Theta\right)^{1/2}}\right]$$

where $$\cos\Theta = \frac{\vec{r}\cdot\vec{r}\,'}{rr'} = \begin{array}{l}\sin\theta\sin\theta'\cos\phi\cos\phi' \\ +\sin\theta\sin\theta'\sin\phi\sin\phi' \\ +\cos\theta\cos\theta'\end{array} = \sin\theta\sin\theta'\cos(\phi-\phi') + \cos\theta\cos\theta'$$

$$\frac{\partial G_D(\vec{r},\vec{r}\,')}{\partial n'} = \hat{n}\cdot\vec{\nabla}_{r'}G_D(\vec{r},\vec{r}\,') = (-\hat{r})\cdot\vec{\nabla}_{r'}G_D(\vec{r},\vec{r}\,') = -\frac{\partial}{\partial r'}G_D(\vec{r},\vec{r}\,')$$

& we need to evaluate this on the surface of the sphere, $r'=R$

$$\left.\frac{\partial G_D}{\partial n'}\right|_{r'=R} = \frac{-1}{4\pi\varepsilon_0}\left[\frac{-\frac{1}{2}\left(2R-2r\cos\Theta\right)}{\left(r^2+R^2-2rR\cos\Theta\right)^{3/2}} - \frac{-\frac{1}{2}\left(2\frac{r^2}{R^2}R-2r\cos\Theta\right)}{\left(r^2+R^2-2rR\cos\Theta\right)^{3/2}}\right]$$

$$= \frac{-1}{4\pi\varepsilon_0}\frac{-R+r^2/R}{\left(r^2+R^2-2rR\cos\Theta\right)^{3/2}} = -\frac{1}{4\pi\varepsilon_0}\frac{1}{R}\frac{(r^2-R^2)}{\left(r^2+R^2-2rR\cos\Theta\right)^{3/2}}$$

an example usage might be to find the potential in a charge-free region of space when the potential is known on the surface of a sphere:

$$\varphi(\vec{r}) = +\frac{1}{4\pi R}\int dS'\,\varphi_s(\vec{r}\,')\,\frac{r^2-R^2}{\left(r^2+R^2-2rR\cos\Theta\right)^{3/2}}$$

$$= \frac{+R}{4\pi}(r^2-R^2)\int d\Omega'\,\varphi(R,\theta',\phi')\,\frac{1}{\left(r^2+R^2-2rR\cos\Theta\right)^{3/2}}$$

e.g. consider a conducting sphere with its upper hemisphere held at a potential V and its lower hemisphere grounded (a small insulating ring separates the hemispheres)

Suppose we want the potential on the z-axis (for $z > R$)

then $$\varphi(z) = \frac{R}{4\pi}(z^2-R^2)\int_0^{2\pi} d\phi' \int_0^{\pi/2} \sin\theta' d\theta' \; V \cdot \frac{1}{(z^2+R^2-2zR\cos\Theta)^{3/2}}$$

$$\cos\Theta = \sin\theta\sin\theta'\cos(\phi-\phi') + \cos\theta\cos\theta' \longrightarrow \cos\theta' \quad \text{for } \theta=0$$

$$\varphi(z) = \frac{R}{4\pi}(z^2-R^2) \cdot 2\pi V \int_0^1 d(\cos\theta') \frac{1}{(z^2+R^2-2zR\cos\theta')^{3/2}}$$

$$\varphi(z) = \frac{V}{2}\left(1+\frac{R}{z}\right)\left(1-\frac{z-R}{\sqrt{z^2+R^2}}\right)$$