(this can be moved to next semester if required)

QUASI-STATIC FIELDS IN CONDUCTORS

We'll consider the case where a magnetic field changes with time, but with a relatively slow rate, such that the finite speed of propagation of electromagnetic disturbances can be neglected.

More specifically we'll consider systems where the wavelength of the magnetic field just outside a conductor is much larger than the size of the conductor, $\lambda >> l$ and thus correspondingly $\nu << c/l$.

...uum $\qquad \vec{\nabla}\cdot\vec{B}=0, \quad \vec{\nabla}\times\vec{H}=0$ in vacuum

...ductor σ

$$\vec{\nabla}\cdot\vec{B}=0, \quad \vec{\nabla}\times\vec{H}=\vec{J}=\sigma\vec{E}$$ for an Ohmic conductor

and we assume the only source of $\vec{E}$-field in the conductor is Faraday induction from the magnetic field

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\cdot\vec{E}=0=\vec{\nabla}\cdot\vec{J} \leftarrow \text{"quasistatic"}$$

e.g. one oscillation of the field takes much longer than the mean free time between collisions in the conduction
$\Rightarrow$ restricted to low frequencies (e.g. lower than infrared for copper)

$[\sigma_{cu} \sim 6\times10^{7}\ \Omega^{-1}m^{-1}]$

combining these equations, we find for the field $\vec{H}$ in the conducting medium

$$\underbrace{\vec{\nabla}\times(\vec{\nabla}\times\vec{H})}_{\vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{H}}_{=0})-\nabla^2\vec{H}} = \sigma\,\vec{\nabla}\times\vec{E} = -\sigma\frac{\partial\vec{B}}{\partial t} = -\sigma\mu\frac{\partial\vec{H}}{\partial t}$$

$$\boxed{\nabla^2\vec{H} = \sigma\mu\frac{\partial\vec{H}}{\partial t}}$$

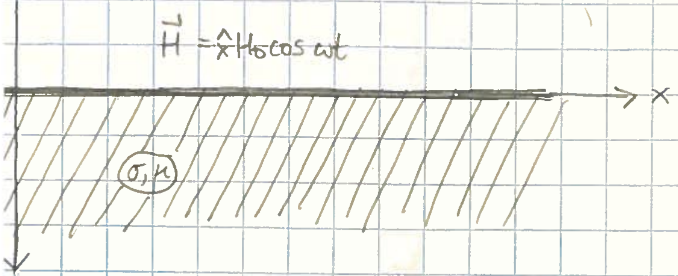
similarly for the electric field

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\mu\frac{\partial}{\partial t}\vec{\nabla}\times\vec{H} = -\mu\sigma\frac{\partial}{\partial t}\vec{E} \xrightarrow[\text{if } \vec{\nabla}\cdot\vec{J}=0]{} \boxed{\nabla^2\vec{E} = \sigma\mu\frac{\partial\vec{E}}{\partial t}}$$

the vector potential, $\vec{A}$, also satisfies the same equation, which we can identify as a "diffusion" equation.

N.B. ignoring the "displacement current" is justified: $\vec{\nabla}\times\vec{H} = \vec{J}+\varepsilon\frac{\partial\vec{D}}{\partial t} = (\sigma+\varepsilon\omega)\vec{E}$ for harmonic freq ω

if $\varepsilon\approx\varepsilon_0 \qquad \omega << \frac{\sigma}{\varepsilon_0} \sim 10^{19}$ Hz for copper effect of a quasi-static magnetic field on a conductor



consider a spatially uniform but time varying magnetic field outside the conductor.

(this is an ohmic conductor, $\vec{J} = \sigma\vec{E}$)

the boundary condition $\hat{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{K}_f$ can be applied assuming no surface current in an imperfect conductor, coupled with $\hat{n}\cdot(\vec{B}_2 - \vec{B}_1) = 0$ we conclude that the magnetic field just inside the conductor is also $\vec{H} = \hat{x} H_0 \cos\omega t$

Away from the boundary we propose $\vec{H}(z>0,t) = \hat{x}\, h(z) \cos\omega t$ with $h(0) = H_0$
For convenience let's use a complex notation

$$\vec{H}(z,t) = \hat{x}\, h(z)\, e^{-i\omega t}$$

and remember to take the real part

$$\nabla^2 \vec{H} = \sigma\mu \frac{\partial \vec{H}}{\partial t} \longrightarrow \left(\frac{d^2}{dz^2} + i\mu\sigma\omega\right) h(z) = 0$$

with $h(z) = e^{ikz} \longrightarrow k^2 = i\mu\sigma\omega \Rightarrow k = \sqrt{i}\sqrt{\sigma\mu\omega} = \pm(1+i)\sqrt{\frac{\sigma\mu\omega}{2}} \equiv \pm(1+i)\frac{1}{\delta}$

defining a quantity with dimension of length, $\delta \equiv \sqrt{\frac{2}{\sigma\mu\omega}}$

then $\vec{H}(z,t) = \hat{x}\, e^{-i\omega t}\left(A e^{iz/\delta} e^{-z/\delta} + B e^{-iz/\delta} e^{z/\delta}\right)$

unacceptable as $z \to \infty$

$$= \hat{x}\, H_0\, e^{-i\omega t + iz/\delta} e^{-z/\delta}$$

take the real part

$$\boxed{\vec{H}(z,t) = \hat{x}\, H_0 e^{-z/\delta} \cos(z/\delta - \omega t)}$$

so the magnetic field penetrates only a distance of order $\delta = \sqrt{\frac{2}{\sigma\mu\omega}}$, the "skin-depth" into the conductor.

The time varying nature of $\vec{H}$ does generate a small electric field and corresponding current:

$$\vec{\nabla}\times\vec{H} = \vec{J} = \sigma\vec{E} \Rightarrow \vec{E} = \frac{\hat{y}}{\sigma}\frac{\partial H_x}{\partial z} = \hat{y}\frac{H_0}{\sigma}\left(-\frac{1}{\delta}e^{-z/\delta}\cos(z/\delta - \omega t) - \frac{1}{\delta}e^{-z/\delta}\sin(z/\delta - \omega t)\right)$$

$$= \hat{y}\left(\frac{-H_0}{\sigma\delta}\right) e^{-z/\delta}\left(\cos(z/\delta - \omega t) + \sin(z/\delta - \omega t)\right) \qquad \left[\sin A + \cos A = -\sqrt{2}\cos(A + 3\pi/4)\right]$$

$$\boxed{\vec{E} = \hat{y}\,\frac{\sqrt{2}}{\sigma\delta} H_0 e^{-z/\delta} \cos(z/\delta - \omega t + 3\pi/4)}$$

nd $\vec{J} = \hat{y} \frac{\sqrt{2}}{\delta} H_0 e^{-z/\delta} \cos\left(z/\delta - \omega t + 3\pi/4\right)$

so that again the current is confined to a region within about δ of the surface – viewed on large scales this would appear to be a "surface current" of magnitude

$$K_y = \int_0^\infty J_y \, dz = \mathrm{Re} \frac{\sqrt{2} H_0}{\delta} e^{-i\omega t} e^{i3\pi/4} \int_0^\infty dz\, e^{-z/\delta} e^{iz/\delta}$$

$$= \mathrm{Re} \frac{\sqrt{2} H_0}{\delta} e^{-i\omega t} \frac{1}{\sqrt{2}}(-1+i) \cdot \frac{\delta}{i-1} \left[e^{-(1-i)z/\delta} \right]_0^\infty$$

$$= \mathrm{Re}\, H_0 e^{-i\omega t} (-1) = \underline{-H_0 \cos\omega t}$$

which matches with the boundary condition

$$H_y^{cond} - H_y^{vac} = K_y \quad \text{with} \quad H_y^{cond} = 0 \;\; \& \;\; H_y^{vac} = H_0 \cos\omega t.$$

-rrents in Ohmic conductors dissipate energy through Ohmic heating at a rate

$$\frac{dW}{dt} = \underbrace{\langle \vec{J} \cdot \vec{E} \rangle}_{\text{time average}} = \sigma \langle E^2 \rangle \qquad \sigma E^2 = \frac{2 H_0^2}{\sigma \delta^2} e^{-2z/\delta} \underbrace{\cos^2\left(\omega t - z/\delta + 3\pi/4\right)}_{\text{average over one cycle} = 1/2}$$

$$\frac{dW}{dt} = \frac{H_0^2}{\sigma \delta^2} e^{-2z/\delta} = \frac{1}{2} \mu \omega H_0^2 e^{-2z/\delta} \;\; \xrightarrow{\text{integrated over the whole conductor}} \;\; \frac{dW}{dt} = \frac{H_0^2}{2\sigma\delta} = \frac{H_0^2}{2\sqrt{2}} \sqrt{\frac{\mu\omega}{\sigma}}$$

next semester we'll discuss how maintaining the magnetic field outside the conductor delivers energy continuously to allow this constant dissipation.

DIFFUSION OR "HEAT" EQUATION

in one dimension $\frac{d^2f}{dx^2} = k\frac{df}{dt}$ with $k>0$

by separation of variables, $f(x,t) = X(x)T(t)$ $\Rightarrow$ $\frac{X''}{X} = \alpha^2 \Big/ \frac{T'}{T} = \gamma \Big/ \alpha^2 = k\gamma$

time equation is trivially integrated $T(t) = Ae^{\gamma t}$

$\rightarrow \gamma > 0,\ \alpha^2 > 0 \quad \Rightarrow X(x) = Be^{\alpha x} + Ce^{-\alpha x}$

$\rightarrow \gamma < 0,\ \alpha^2 < 0 \quad \Rightarrow X(x) = B\sin\tilde{\alpha}x + C\cos\tilde{\alpha}x$ $\leftarrow$ suggests Fourier techniques

---

e.g. consider a function which at $t=0$ is $f(x,0) = e^{-x^2/x_0^2}$

$\rightarrow$ how does this function evolve with t if it obeys the diffusion eqn?

$$f(x,0) = e^{-x^2/x_0^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp\, g(p) e^{ipx}$$

$$\Rightarrow g(p) = \int_{-\infty}^{\infty} dx\, e^{-x^2/x_0^2} e^{-ipx} = \sqrt{\pi}\,x_0\, e^{-p^2x_0^2/4}$$

$$= \frac{x_0}{\sqrt{4\pi}}\int_{-\infty}^{\infty} dp\, e^{-p^2x_0^2/4} e^{ipx}$$

but each piece e^{ipx} is associated with a time dependence $e^{+\gamma t} = e^{-p^2/k\cdot t}$

$$\text{so } f(x,t) = \frac{x_0}{\sqrt{4\pi}}\int_{-\infty}^{\infty} dp\, e^{-p^2x_0^2/4} e^{ipx} e^{-p^2/k\, t}$$

$$= \frac{x_0}{\sqrt{4\pi}}\cdot\int_{-\infty}^{\infty} dp\, e^{-p^2/4\,(x_0^2 + 4t/k)} e^{ipx}$$

but we can evaluate this integral

$$f(x,t) = \frac{x_0}{\sqrt{4\pi}}\cdot\frac{\sqrt{4\pi}}{\sqrt{x_0^2 + 4t/k}}\, e^{-x^2/(x_0^2+4t/k)}$$

$$f(x,t) = \frac{1}{\sqrt{1+\frac{4t}{kx_0^2}}}\, e^{-\frac{x^2}{x_0^2+4t/k}}$$

which is getting broader over time — "diffusing" out in oscillatory time dependence, $\gamma = i\omega$

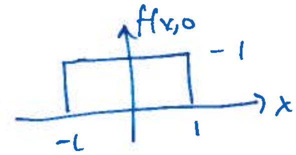
then $\alpha^2 = ik\omega$ $\quad\quad$ $\alpha = \pm \frac{1+i}{\sqrt{2}} \sqrt{k\omega}$ $\quad$ $\Rightarrow$ oscillatory and damped behaviour

$$= \pm (1+i)\rho$$

$$e^{\alpha x} = e^{-\rho} e^{-i\rho} \quad \text{etc...}$$

DIFFUSION OF A TOP-HAT DISTRIBUTION

Suppose a system obeys $\frac{\partial^2 f}{\partial x^2} = k \frac{\partial f}{\partial t}$ $(k>0)$ with initial distribution $f(x,0) = \begin{cases} 1 \text{ for } |x|<1 \\ 0 \text{ for } |x|>1 \end{cases}$



it's easy to show (see HW1) that $f(x,0)$ has a fourier representation

$$f(x,0) = \frac{2}{\pi} \int_0^\infty dp \, \frac{\sin p}{p} \cos px$$

for $f \sim \cos px$, $\frac{\partial^2 f}{\partial x^2} = -p^2 f$

$f \sim e^{\gamma t}$ $\frac{\partial f}{\partial t} = \gamma f$

$$\left.\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= -p^2 f \\ \frac{\partial f}{\partial t} &= \gamma f \end{aligned}\right\} \quad -p^2 = k\gamma \quad \Rightarrow \quad \gamma = -p^2/k$$

$$\Rightarrow f(x,t) = \frac{2}{\pi} \int_0^\infty dp \, \frac{\sin p}{p} \cos px \cdot e^{-p^2 t/k}$$

$$\sin p \cos px = \frac{1}{2}\sin(p-px) + \frac{1}{2}\sin(p+px)$$

$$f(x,t) = \frac{1}{\pi} \int_0^\infty dp \, \frac{\sin p(1-x)}{p} e^{-p^2 t/k} + (x \to -x)$$

$$q = p(1-x)$$

$$= \frac{1}{\pi} \int_0^\infty dq \, \frac{\sin q}{q} e^{-q^2 \cdot \frac{t}{k(1-x)^2}} + (x \to -x)$$

$$= \frac{1}{2} \operatorname{erf}\left(\sqrt{\frac{k}{t}} \, \frac{1-x}{2}\right) + (x \to -x)$$

$$\boxed{f(x,t) = \frac{1}{2} \operatorname{erf}\left(\sqrt{\frac{k}{t}} \, \frac{1-x}{2}\right) + \frac{1}{2} \operatorname{erf}\left(\sqrt{\frac{k}{t}} \, \frac{1+x}{2}\right)}$$

INTEGRAL REPRESENTATIONS OF THE ERROR FN

start with $\int_0^\infty dt\, e^{-t^2} \cos 2xt = \frac{1}{2} \mathrm{Re} \int_{-\infty}^{\infty} dt\, e^{-t^2} e^{i2xt} = \frac{1}{2} \mathrm{Re} \int_{-\infty}^{\infty} dt\, e^{-(t-ix)^2} e^{-x^2}$

$$= \frac{1}{2} \mathrm{Re}\, e^{-x^2} \int_{-\infty}^{\infty} d\bar{t}\, e^{-\bar{t}^2} = \frac{\sqrt{\pi}}{2} e^{-x^2}$$

$$\int_0^\infty dt\, e^{-t^2} \cos 2xt = \frac{\sqrt{\pi}}{2} e^{-x^2} \qquad (I_1)$$

now note that $\int_0^y dx \cos 2xt = \left[\frac{\sin 2xt}{2t}\right]_0^y = \frac{\sin 2yt}{2t}$

so integrating both sides of (I_1):

$$\int_0^\infty dt\, e^{-t^2} \frac{\sin 2yt}{2t} = \frac{\sqrt{\pi}}{2} \int_0^y dx\, e^{-x^2}$$

& changing variables to $z = 2yt$

$$= \frac{1}{2y} \int_0^\infty dz\, e^{-z^2/4y^2} \frac{\sin z}{z/y} = \frac{1}{2} \int_0^\infty dz\, e^{-z^2/4y^2} \frac{\sin z}{z}$$

$$\Rightarrow \int_0^y dx\, e^{-x^2} = \frac{1}{\sqrt{\pi}} \int_0^\infty dz\, e^{-z^2/4y^2} \frac{\sin z}{z}$$

& thus $\mathrm{erf}(y) \equiv \frac{2}{\sqrt{\pi}} \int_0^y dx\, e^{-x^2} = \frac{2}{\pi} \int_0^\infty dx\, e^{-x^2/4y^2} \frac{\sin x}{x}$ Jackson 5.175