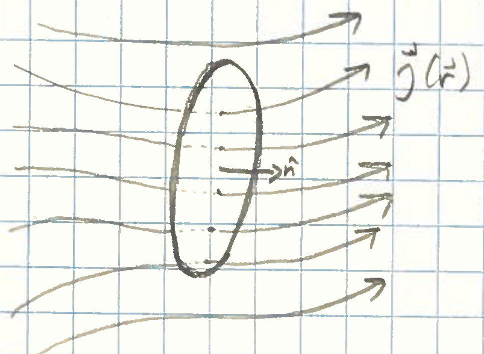
STEADY CURRENT



The flow of electrical current varying over space can be described by a current density

$$\vec{j}(\vec{r},t) = \rho(\vec{r},t)\,\vec{v}(\vec{r},t)$$

and the usual definition of current as charge/unit time follows as

$$I = \frac{dQ}{dt} = \int_S d\vec{S}\cdot\vec{j}$$

Conservation of charge implies the continuity eqn: consider S to be a CLOSED surface, then

$$-\frac{dQ}{dt} = -\frac{\partial}{\partial t}\int_V d^3\vec{r}\,\rho(\vec{r}) = \int_S d\vec{S}\cdot\vec{j} = \int_V d^3\vec{r}\;\vec{\nabla}\cdot\vec{j} \quad \text{(by the divergence theorem)}$$

loss of charge from V

& thus $$\underline{\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0} \quad \text{"continuity" eqn.}$$

Steady currents IN MATTER

Experimentally we find that in many materials, the following relation holds between the charge density and an applied electric field:

$$\vec{j} = \sigma\vec{E} \qquad \text{"Ohm's law"}$$

σ = conductivity – a property of the material

At first sight, this is a suprising result – the force on a charge q due to $\vec{E}$ is $q\vec{E}$, but $\vec{j} \propto \vec{v}$, the velocity of the charge, which seems to imply that force is proportional to velocity, in contradiction to Newton's 2nd law.

The explanation is that the electrostatic force is not the only one acting on the charges – there is also an effective drag force due to repeated collisions with other particles in the material

A simple, but illustrative classical model is due to Drude:

consider electrons in a metal feeling an electrostatic force $\vec{F}_E = -e\vec{E}$

+ a drag force $\vec{F}_d = -m\vec{v}/\tau$

where τ is the average time between collisions

then Newton's 2^{nd} law reads

$$m\frac{d\vec{v}}{dt} = -e\vec{E} - \frac{m\vec{v}}{\tau}$$

e.g. in the direction of a constant $\vec{E}$: $\displaystyle m\frac{dv}{dt} = -eE - \frac{mv}{\tau}$

$$\frac{dv}{\tau\frac{e}{m}E + v} = -\frac{dt}{\tau}$$

$$\ln\left[\frac{e\tau}{m}E + v\right] = -t/\tau + c$$

$$v + \frac{e\tau}{m}E = Ae^{-t/\tau}$$

$$v(t) = -\frac{e\tau}{m}E + Ae^{-t/\tau}$$

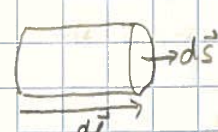
So for times much longer than τ $\quad v \to -\frac{e\tau}{m}E \quad$ "the drift velocity"

$$\Rightarrow j = \rho v = n(-e)v = \left(\frac{ne^2\tau}{m}\right)E$$

& Ohm's law results

n: number density

An important idealisation is when the current is localised in a wire where the flow $\vec{j}$ is everywhere parallel to the wire



then $\int d^3\vec{r}\, \vec{j} = \int d\vec{s}\cdot d\vec{l}\; \vec{j}$

$= \int d\vec{l} \int d\vec{s}\cdot\vec{j} = I\int d\vec{l}$

e.g. $\int d^3\vec{r}\; \vec{j}\times\vec{c} = I\int d\vec{l}\times\vec{c}$

for steady currents, there is no build up of charge over time, $\frac{d\rho}{dt}=0$

and in an ohmic material we have

$\vec{\nabla}\cdot\vec{j}=0$ "steady current" $\vec{j}=\sigma\vec{E}$ "Ohm's law" $\vec{\nabla}\times\vec{E}=\vec{0}$ [an "emf" is required to maintain $\vec{j}\neq\vec{0}$]

for position independent conductivity

$$0=\sigma\vec{\nabla}\cdot\vec{E}=\sigma\rho(\vec{r})/\varepsilon_0 \Rightarrow \rho(\vec{r})=0$$ – there is no volume distribution of charge

(There can be a distribution of charge on the surface of the conductor)

Laplace's eqn will hold, $\nabla^2\varphi=0$.

Suppose we have an ohmic conductor which has two faces held at potential V_A and V_B respectively

If no current flows out of the sides, we can define the RESISTANCE of the object as

$$R=\frac{V_A-V_B}{I}$$

but $V_A-V_B=\int_B^A\frac{d\varphi}{dl}dl=-\int_A^B\frac{d\varphi}{dl}dl=\int_A^B\vec{E}\cdot d\vec{l}$

& $I=\int_S d\vec{S}\cdot\vec{j}=\sigma\int_S d\vec{S}\cdot\vec{E}$

$$\Bigg\}\ R=\frac{1}{\sigma}\frac{\int\vec{E}\cdot d\vec{l}}{\int\vec{E}\cdot d\vec{S}}$$

e.g. consider a wire of length L with uniform, but arbitrary cross-section of area A. Find the resistance:

Imagine setting up a parallel plate capacitor with plate separated by L, and the plates at potential V_A, V_B respectively. The $\vec{E}$-field between them is uniform and of magnitude $(V_A-V_B)/L$.

If we place the wire between the plates & in contact with the plates, the $\vec{E}$ field will not change because Laplace's eqn still holds. $\vec{E}$ is still uniform & thus so is $\vec{j}$.

$$\Rightarrow \int d\vec{S}\cdot\vec{j}=\sigma\int d\vec{S}\cdot\vec{E}=\sigma\cdot\frac{V_A-V_B}{L}A \quad\Rightarrow\quad \underline{R=\frac{1}{\sigma}L/A}.$$

optional - skip it behind

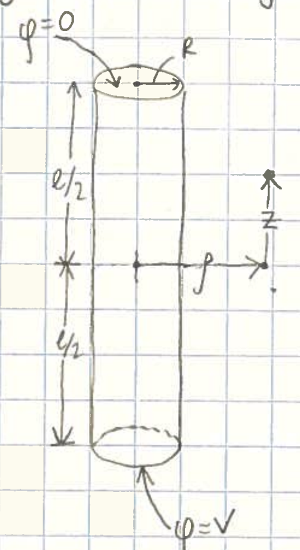
A tricky question is "where are the charges which maintain $\vec{E} \neq \vec{0}$ in an ohmic material ?"

$$(\vec{E} = \vec{J}/\sigma)$$

we already showed that it can't be a volume distribution in the material, nor can it be outside the material - the only place left is on the <u>SURFACE</u> of the material.

This would also imply that there is a field outside the material.

e.g. consider a very long straight wire with circular cross-section



Laplace's eqn holds inside & out.

Inside : $\varphi_{in} = -E_0 z$, $E_0 = V/\ell$

outside : $\varphi_{out} = A z \ln(\rho/L)$ L = an unknown length scale

[a $\ln \rho/L$ term, without the z would correspond to a net charge on the wire]

$$\varphi_{in}(\rho=R) = \varphi_{out}(\rho=R)$$

$$-E_0 = A \ln R/L \quad \Rightarrow A = -V/\ell \, \frac{1}{\ln R/L}$$

$$\varphi_{out}(\rho, z) = -V \cdot \frac{z}{\ell} \, \frac{\ln(\rho/L)}{\ln(R/L)}$$

this derivation doesn't prove it, but the correct choice for L is the length of the wire, ℓ

$$\varphi_{out}(\rho, z) = -V \frac{z}{\ell} \frac{\ln \rho/\ell}{\ln R/\ell} \quad \text{valid for } z, \rho \ll \ell.$$

$$\Rightarrow \vec{E}_{out} = \frac{V}{\ell}\left[\frac{\ln \rho/\ell}{\ln R/\ell}\hat{z} + \frac{z/\rho}{\ln R/\ell}\hat{\rho}\right] \quad \& \quad \vec{E}_{in} = V/\ell \, \hat{z}$$

the surface charge density is $\sigma(z) = \varepsilon_0\left(\hat{\rho}\cdot\vec{E}_{out}(\rho=R, z) - \hat{\rho}\cdot\vec{E}_{in}(\rho=R, z)\right)$

$$\sigma(z) = \varepsilon_0 \frac{V}{\ell} \frac{z/R}{\ln(R/\ell)} \qquad \left[\int_{-\ell/2}^{\ell/2} \sigma(z)\,dz = 0 \checkmark\right]$$