HW 1 - THE MATHEMATICS OF WAVES

All problems due Wed 27th Jan.

1. Checking the solution of the wave equation

In lecture we derived the solution,

$$\psi(\mathbf{r},t) = -\frac{1}{4\pi}\int d^3\mathbf{r}'\,\frac{1}{|\mathbf{r}-\mathbf{r}'|}\,f\left(\mathbf{r}',t-\tfrac{1}{c}|\mathbf{r}-\mathbf{r}'|\right), \tag{1}$$

of the inhomogeneous wave equation,

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\psi(\mathbf{r},t) = f(\mathbf{r},t). \tag{2}$$

Show that Eqn. 1 solves Eqn. 2 by direct substitution.

2. Another derivation of the Green function

In lecture we derived the Green function,

$$G(\mathbf{r},t;\mathbf{r}',t) = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\,\delta\Big(t-t'\pm\tfrac{1}{c}|\mathbf{r}-\mathbf{r}'|\Big) \tag{3}$$

for the wave equation,

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]G(\mathbf{r},t;\mathbf{r}',t) = -4\pi\,\delta(\mathbf{r}'-\mathbf{r})\,\delta(t'-t).$$

In this problem we'll explore an alternative derivation using Fourier methods.

Firstly, realize that we can rewrite the defining equation using variables $\boldsymbol{\rho} = \mathbf{r}-\mathbf{r}'$ and $\tau = t-t'$, as

$$\left[\nabla_\rho^2 - \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2}\right]G(\boldsymbol{\rho},\tau) = -4\pi\,\delta(\boldsymbol{\rho})\,\delta(\tau).$$

and eliminate the explicit time-dependence by using a Fourier transform of G,

$$G(\boldsymbol{\rho}, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, g(\boldsymbol{\rho}, \omega) \, e^{-i\omega\tau},$$

and the Fourier representation of the delta function,

$$\delta(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, e^{-i\omega\tau},$$

to obtain the equation

$$\left[\nabla_\rho^2 + k^2\right] g(\boldsymbol{\rho}, \omega) = -4\pi \, \delta(\boldsymbol{\rho}), \tag{4}$$

where $k = \omega/c$.

By writing a Fourier representation of $g(\boldsymbol{\rho}, \omega)$,

$$g(\boldsymbol{\rho}, \omega) = \frac{1}{(2\pi)^3} \int d^3\mathbf{k}' \, \tilde{g}(\mathbf{k}', \omega) \, e^{i\mathbf{k}'\cdot\boldsymbol{\rho}}, \tag{5}$$

solve Eqn. 4 to show that $\tilde{g}(\mathbf{k}', \omega) = -4\pi \left(|\mathbf{k}|^2 - |\mathbf{k}'|^2\right)^{-1}$. Perform the angular integration in Eqn. 5 to leave an integral

$$\frac{1}{i\pi\rho} \int_{-\infty}^{\infty} dk' \, \frac{k'}{k'^2 - k^2} e^{ik'\rho}. \tag{6}$$

Introduce a regulator by replacing k^2 with $k^2 + i\epsilon$, considering ϵ to be small and positive. Identify the singularity structure of the integrand and by choosing a suitable contour in the complex k'-plane compute the integral, and show that you recover one of the two sign choices in Eqn. 3 in the limit $\epsilon \to 0$ – the other sign choice follows if ϵ is considered small and negative.

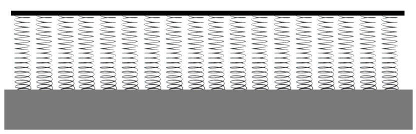
3. A simple example of dispersion - waves on a string

(a) A string of mass per unit length μ is held under tension, T. The unperturbed string lies along the x-axis. Show that transverse waves on the string satisfy

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

with $c^2 = T/\mu$ (you may look this up in a book if you're unfamiliar with the construction). Monochromatic waves, $y(x,t) = A\, e^{i(kx-\omega t)}$ (take the real part to get physical waves), then have dispersion relation $\omega = kc$ and the phase and group velocities are identical and equal to c.

(b) Now consider the case shown in the figure, where the string is attached to a (continuous) distribution of springs (obeying Hooke's law) that have their other end fixed to a wall. When the string is unperturbed, the springs are at their equilibrium length.



If the "spring constant per unit length (in x)" is σ, show that waves on the string now must satisfy

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - \omega_s^2 y,$$

with $\omega_s^2 = \sigma/\mu$. Find the dispersion relation in this case and consider the phase and group velocities. What form do solutions to this equation with $\omega < \omega_s$ take ?

(c) Suppose the original tensioned string is immersed in a viscous fluid, so that there is a drag force proportional to speed. Show that waves on the string now must satisfy

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - \beta \frac{\partial y}{\partial t}.$$

Find the dispersion relation and consider the form of solutions for $x > 0$ when β is small. How to these waves differ from those with $\omega < \omega_s$ in part (b) ?