HW 2 - MAXWELL'S EQUATIONS

All problems due Wed 3rd Feb.

1. $\varphi, \mathbf{A}$ and charge conservation

Check that the inhomogeneous wave equations for φ and $\mathbf{A}$ are compatible with conservation of charge: (a) in Lorenz gauge (b) in Coulomb gauge.

2. Maxwell's equations and magnetic charge

Suppose we tried to generalize the Maxwell equations to allow for "magnetic charge", q_m, as well as "electric charge", q_e, then we might try

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho_e/\epsilon_0 \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = \mu_0 \rho_m \tag{1}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{\epsilon_0 c^2}\mathbf{J}_m - \frac{\partial \mathbf{B}}{\partial t} \qquad \boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}_e + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}, \tag{2}$$

and for the force per unit volume on a "charge" distribution,

$$\mathbf{f} = \big(\rho_e \mathbf{E} + \mathbf{J}_e \times \mathbf{B}\big) + \big(\rho_m \mathbf{B} - \tfrac{1}{c^2}\mathbf{J}_m \times \mathbf{E}\big). \tag{3}$$

(a) Check that these Maxwell equations ensure that electric and magnetic charge are separately conserved.

(b) Consider the following "duality" transformation of the fields and charges:

$$\begin{bmatrix} \mathbf{E}' \\ c\mathbf{B}' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ c\mathbf{B} \end{bmatrix} \tag{4}$$

$$\begin{bmatrix} c\,q_e' \\ q_m' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} c\,q_e \\ q_m \end{bmatrix}. \tag{5}$$

Show that the Maxwell equations and the force density are invariant under these transformations. Show that $\mathbf{E} \times \mathbf{B}$ and $\frac{1}{2}\epsilon_0 \mathbf{E} \cdot \mathbf{E} + \frac{1}{2\mu_0}\mathbf{B} \cdot \mathbf{B}$ are also invariant.

3. Using the Maxwell Stress Tensor

(a) Consider the electric field generated jointly by two point charges, $\pm Q$, separated by a distance d. Find an expression for the Maxwell stress tensor on the midplane between the charges and use it to find the rate at which the electric field delivers momentum across the midplane. Show that this rate of delivery agrees with the Coulomb force of one charge on the other.

(b) A cube of dimension $L \times L \times L$ finds itself in an electric field that is everywhere in the z-direction. The field has a constant magnitude E_1 at the bottom and side faces and a constant magnitude E_2 at the top face. Find the force that the field exerts on the cube (noting that we don't know anything about the charge distribution within the cube).

4. Energy flow into a current carrying wire

Calculate the Poynting vector at the surface of a resistive wire of circular cross-section that is carrying a steady current. Show that the integral of the Poynting vector over the wire's surface is equal to the power dissipated in the wire calculated using Ohm's Law.