HW 3 - PLANE WAVES

All problems due Feb 17th.

1. Energy flow in reflection and refraction of plane waves

Consider a monochromatic plane-wave with electric and magnetic fields,

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \qquad \mathbf{H}(\mathbf{r},t) = \mathbf{H}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{1}$$

where in a medium of (real) refractive index, $n = \sqrt{(\epsilon/\epsilon_0)(\mu/\mu_0)}$ and (real) impedance $Z = \sqrt{\mu/\epsilon}$,

$$\mathbf{k} = n\frac{\omega}{c}\hat{\mathbf{k}}, \quad \mathbf{k}\cdot\mathbf{E} = 0, \quad Z\mathbf{H} = \hat{\mathbf{k}}\times\mathbf{E}. \tag{2}$$

A plane wave traveling in a semi-infinite medium (ϵ_1, μ_1) encounters a plane boundary with a semi-infinite medium (ϵ_2, μ_2) and undergoes transmission and reflection. We may define the *energy reflection coefficient* as

$$R = \left|\frac{\langle \mathbf{S}_r\rangle\cdot\hat{\mathbf{n}}}{\langle \mathbf{S}_i\rangle\cdot\hat{\mathbf{n}}}\right| \tag{3}$$

where $\mathbf{S}_{i(r)}$ is the time-averaged Poynting vector for the incident(reflected) wave and $\hat{\mathbf{n}}$ is the unit normal to the plane boundary.

Show that $R = |r|^2$ where r is the Fresnel reflection amplitude defined in the notes.

The *energy transmission coefficient* may be similarly defined as

$$T = \left|\frac{\langle \mathbf{S}_t\rangle\cdot\hat{\mathbf{n}}}{\langle \mathbf{S}_i\rangle\cdot\hat{\mathbf{n}}}\right|. \tag{4}$$

Find expressions for T in terms of Z_1, Z_2 and the angles of incidence and transmission, in the case of "mp" and "ep" orientations as defined in the notes.

Check that energy is conserved in the reflection/transmission process by showing that $R + T = 1$ for both "mp" and "ep".

2. Total internal reflection

Consider the "mp" wave incident in medium 1 on a plane boundary with medium 2 that we considered on page 27 of the notes. Show that the wave transmitted into medium 2 when $\sin\theta_1 > n_2/n_1$ has magnetic field

$$\mathbf{H}_2(\mathbf{r},t) = \frac{1}{\mu_2\,\omega}\mathbf{k}_2 \times \mathbf{E}_2(\mathbf{r},t), \tag{5}$$

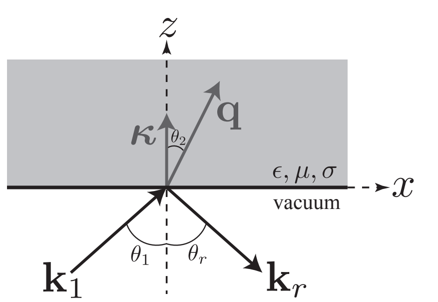
with $\mathbf{k}_2 = \frac{\omega}{c}\left(n_1 \sin\theta_1\,\hat{\mathbf{x}} + i\sqrt{n_1^2\sin^2\theta_1 - n_2^2}\,\hat{\mathbf{z}}\right)$, and show that the time-averaged flow of energy across the boundary is zero ($\hat{\mathbf{z}}\cdot\langle\mathbf{S}_2\rangle = 0$).

[*Notice that while* $\mathbf{k}_2\cdot\mathbf{E}_2 = 0$*, it does not follow that* $\mathbf{k}_2\cdot\mathbf{E}_2^* = 0$]

By considering $\hat{\mathbf{x}}\cdot\langle\mathbf{S}_2\rangle = 0$, show that in medium 2 energy does flow in the $\hat{\mathbf{x}}$ direction in a small layer close to the boundary.

3. Refraction into a good conductor

Consider plane waves incident in vacuum onto a plane boundary with a good conductor – the diagram defines some variables. The wavevector in the conductor must be complex, $\mathbf{k}_2 = \mathbf{q} + i\boldsymbol{\kappa}$, to allow for the Ohmic energy loss.



(a) Explain why $\boldsymbol{\kappa}$ must point in the $\hat{\mathbf{z}}$-direction and using an argument about the phase of the incident, reflected and transmitted waves, show that the angle of reflection is equal to the angle of incidence. Show that Snell's law has an analogue here: $q\sin\theta_2 = k_1\sin\theta_1$.

(b) Show that κ is a solution of

$$\kappa^4 + \kappa^2 k_1^2\big(\mu_r\epsilon_r - \sin^2\theta_1\big) - \frac{1}{4}\big(\mu\sigma\omega\big)^2 = 0,$$

and that for a good conductor, $\kappa \to \frac{1}{\delta(\omega)}$ where $\delta(\omega)$ is the skin-depth.

(c) Show that the refracted wave to a good approximation propagates along the $\hat{\mathbf{z}}$ direction in a good conductor, independent of the angle of incidence.

4. Radiation pressure on a perfect conductor

A plane-wave with electric field $\mathbf{E}(\mathbf{r},t) = \hat{\mathbf{y}}E_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ where $\mathbf{k} = -k\cos\theta\hat{\mathbf{x}} + k\sin\theta\hat{\mathbf{z}}$ is incident on a perfect conductor which occupies the half-space $x < 0$. (E_0 is real in this case).

Find the time-averaged pressure exerted on the conductor, $P = \epsilon_0 E_0^2 \cos^2\theta$,

(a) by evaluating the Lorentz force on the surface currents generated in the conductor, and

(b) by evaluating the change in linear momentum carried by the incident and reflected waves.

[*Hint:* the force per unit area on a surface current density $\mathbf{K}$ having fields $\mathbf{B}_1$, $\mathbf{B}_2$ on its two sides is $\frac{1}{2}\mathbf{K}\times(\mathbf{B}_1+\mathbf{B}_2)$ - note the similarity in form to the force on a surface charge distribution]