HW 4 - WAVEGUIDES AND CAVITIES

All problems due Wed 2nd March.

1. A rectangular cavity

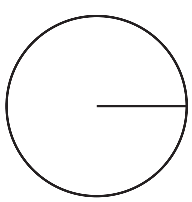
Consider a rectangular cavity of dimensions $a \times b \times d$ featuring a field oscillating in the TM_{011} mode (i.e. $k_z = 0$ and the longest wavelength behavior in the x and y directions).

(a) Find the charge density and surface current density everywhere on the walls of the cavity.

(b) Check that $\boldsymbol{\nabla} \cdot \mathbf{K} + \frac{\partial \sigma}{\partial t} = 0$ everywhere.

2. A circular waveguide with a baffle

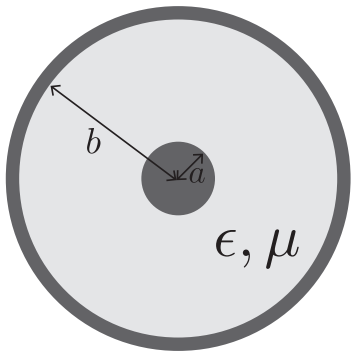
A perfectly conducting waveguide has a circular cross-section of radius R. An infinitesimally thin metal baffle plate is inserted into the interior so it runs from the center out to a point on the circumference (see the diagram).



Show that the presence of the baffle increases the lowest cutoff frequency for TM modes, but decreases the lowest cutoff frequency for TE modes.

3. A TEM mode in a coaxial cable

A transmission line consists of two concentric circular cylinders of metal with conductivity σ as shown, with the gap between them filled with a uniform lossless dielectric (ϵ, μ).



Unlike in the case of a cylindrical waveguide (as discussed on page 48 of the notes), TEM modes can propagate along this line. A TEM mode propagating in the $\hat{\mathbf{z}}$ direction has $E_z = H_z = 0$ and $Z\mathbf{H}_\perp = \hat{\mathbf{z}} \times \mathbf{E}_\perp$.

$\mathbf{E}_\perp$ must satisfy $\boldsymbol{\nabla}_\perp \times \mathbf{E}_\perp = 0$ and $\boldsymbol{\nabla}_\perp \cdot \mathbf{E}_\perp = 0$, so we can define a potential, $\mathbf{E}_\perp = -\boldsymbol{\nabla}_\perp \varphi$, which must be a solution of the two-dimensional Poission equation, $\nabla_\perp^2 \varphi = 0$.

(a) Convince yourself that $\varphi(\rho) = A \log \rho$ is the only azimuthally symmetric solution possible so $\mathbf{E}_\perp = -A\frac{1}{\rho}\hat{\boldsymbol{\rho}}$. Show that for any value of z, the following monochromatic waves satisfy the conditions needed to be an acceptable TEM wave,

$$\begin{aligned} \mathbf{E}(\rho, \phi, z, t) &= Z H_0(z) \frac{a}{\rho} e^{i(kz-\omega t)} \hat{\boldsymbol{\rho}} \\ \mathbf{H}(\rho, \phi, z, t) &= H_0(z) \frac{a}{\rho} e^{i(kz-\omega t)} \hat{\boldsymbol{\phi}}, \end{aligned} \tag{1}$$

but at this stage we will not determine how $H_0(z)$ should depend upon z – we'll determine this by considering the energy losses in the walls.

(b) Show that, assuming the forms above, the time-averaged power flow along the line is

$$\langle P \rangle = Z\, \pi a^2 \left|H_0(z)\right|^2 \log b/a \tag{2}$$

(c) Since the metal walls are imperfect conductors, some energy is lost in them by Ohmic heating. Recall that in the notes we showed that

$$\frac{d\langle P_{\text{loss}} \rangle}{dA} = \frac{1}{2\sigma\delta} \left|\mathbf{K}_{\text{eff}}\right|^2, \tag{3}$$

where $\mathbf{K}_{\text{eff}}$ is the effective surface current (the surface current we'd have if the conductor were perfect).

Show that the power lost in this way per unit length of line is

$$\frac{d\langle P_{\text{loss}}\rangle}{dz} = \frac{\pi}{\sigma\delta}\big|H_0(z)\big|^2 a\Big(1 + \frac{a}{b}\Big), \tag{4}$$

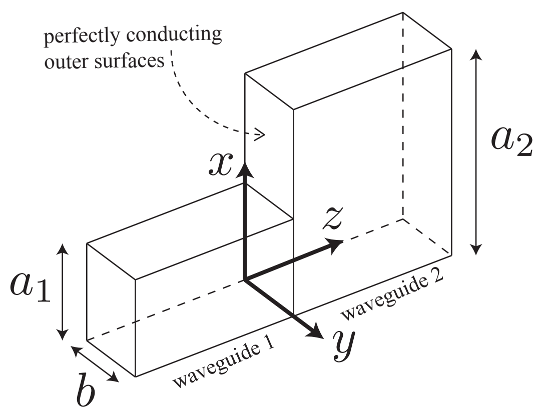
(d) Show that the transmitted power is attenuated as $\langle P\rangle(z) = \langle P\rangle(0)\, e^{-\beta z}$ with

$$\beta = \frac{1}{\sigma\delta}\frac{1}{Z}\frac{a^{-1} + b^{-1}}{\log b/a} \tag{5}$$

[*for you to ponder:* if we plug Eq. 1 into Maxwell's equations inside the waveguide, we'll get an equation for $H_0(z)$ which isn't compatible with the exponential damping shown above – somewhere we built in an inconsistent assumption ...]

4. Joining two waveguides

Two rectangular waveguides meet as shown in the figure. Waveguide 1 continues to $z \to -\infty$ and waveguide 2 continues to $z \to \infty$.



If a TE_{10} mode ($H_z \propto \cos\left[\pi x/a_1\right]$) propagates in waveguide 1 in the $+\hat{\mathbf{z}}$ direction, find the amplitudes of the various modes excited in waveguide 2.