HW 5 - RADIATION

All problems due Wed 30th March.

1. A classical model of the Zeeman effect

Suppose we model the classical single electron atom as an infinitely heavy point nucleus of charge $+e$ to which an electron of mass m and charge $-e$ is bound by a harmonic restoring force (i.e. a spring). Explain why in this model of the atom, the position of the electron with respect to any stationary equilibrium position, $\mathbf{s}$, must satisfy

$$m\ddot{\mathbf{s}} = -m\omega_0^2\mathbf{s}. \tag{1}$$

Now suppose a uniform magnetic field is applied – explain why the appropriate equation of motion is now

$$m\ddot{\mathbf{s}} = -m\omega_0^2\mathbf{s} + (-e)\dot{\mathbf{s}} \times \mathbf{B}. \tag{2}$$

If the magnetic field is in the $\hat{\mathbf{z}}$ direction, defining the Larmor frequency $\omega_L = \frac{eB}{2m}$, find equations for the cartesian components of $\mathbf{s}$ assuming harmonic oscillation, $\mathbf{s}(t) = \mathbf{A}e^{-i\omega t}$. Solve these equations for frequencies $\omega \gg \omega_L$ to obtain

$$\mathbf{s}_0(t) = A_0\,\hat{\mathbf{z}}\,e^{-i\omega_0 t} \tag{3}$$

$$\mathbf{s}_\pm(t) = A_\pm \left(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}\right) e^{-i(\omega_0 \pm \omega_L)t} \tag{4}$$

Show that in the case of oscillation purely in the "0" mode, the radiation field produced in linearly polarized and has an angular distribution of power $\propto \sin^2\theta$.

Show that in the case of oscillation in either of the $\pm$ modes, the radiation field is circularly polarized when measured at a point on the z-axis, and that the angular distribution of power is $\propto (1 + \cos^2\theta)$.

2. No dipole radiation

Consider a collection of N moving electrons. There are no other charges nearby to exert external forces on the system of electrons. Show that the collection does not produce any electric or magnetic dipole radiation.

3. An oscillating point charge

A single particle of charge q moves along the z-axis according to $z(t) = b \sin \omega t$.

(a) Show that the fields due to the electric dipole moment oscillate with frequency ω and compute the power radiated as E1.

(b) Show that the fields due to the electric quadrupole oscillate with frequency 2ω and compute the power radiated as E2.

4. An uncharged rotor

Two equal and opposite charges are attached to the ends of a rod of length $2L$. The center of the rod lies at the origin of the coordinate system and the rod rotates counterclockwise in the $x - y$ plane with angular speed ω. At $t = 0$ the rod points in the $\hat{\mathbf{x}}$ direction.

(a) Show that the radiation electric field is (the real part of)

$$\mathbf{E}_{\text{rad}}(r, \theta, \phi; t) = \mathbf{p}(t = 0) \frac{\mu_0 \omega^2}{4\pi} \big(\cos\theta \, \hat{\boldsymbol{\theta}} + i \hat{\boldsymbol{\phi}} \big) \frac{e^{i(kr - \omega t + \phi)}}{r} \tag{5}$$

(b) What polarization does the radiation have at points on the (i) x-axis, (ii) y-axis, (iii) z-axis ?

(c) Find the time-averaged rate at which energy is radiated per unit solid angle and the total rate at which energy is radiated to infinity.

5. A charged rotor

Two identical point charges, q, are attached to the ends of a rod of length $2L$. The center of the rod lies at the origin of the coordinate system and the rod rotates counterclockwise in the $x-y$ plane with angular speed ω.

(a) Show that the electric dipole moment of the rotor is zero.

(b) Show that rotor has a non-zero magnetic dipole moment, but that there is no magnetic dipole radiation

(c) Show that the (cartesian) electric quadrupole tensor, $Q_{ij}(t)$, is

$$\frac{1}{2}qL^2\begin{bmatrix}1+\cos 2\omega t & \sin 2\omega t & 0\\ \sin 2\omega t & 1-\cos 2\omega t & 0\\ 0 & 0 & 0\end{bmatrix}. \tag{6}$$

(d) Show that the time averaged angular distribution of radiated power due to the electric quadrupole is

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{\mu_0}{4\pi}\frac{2q^2\omega^6L^4}{\pi c^3}\left(1-\cos^4\theta\right) \tag{7}$$

6. A spherical 'antenna'

Two halves of a spherical metallic shell of radius, R, and infinite conductivity are separated by a small insulating gap. An alternating potential is applied between the two halves of the sphere so that the potentials are $\pm V_0\cos\omega t$.

In the long-wavelength limit, find the radiation fields, the angular distribution of radiated power and the total power radiated from the sphere.

[*Hint:* Find the potential outside the sphere in the static case where the upper hemisphere is at potential $+V_0$ and the lower hemisphere is at $-V_0$. What is the lowest multipole present in this potential?]