HW 7 - SPECIAL RELATIVITY IN ELECTROMAGNETISM

All problems due April 20th.

1. Motion in a uniform electric field

A particle of charge q and mass m, initially at rest, is acted upon by a uniform electric field of magnitude E directed along the x-axis. Find an expression for the position of the particle as a function of time.

What is the asymptotic speed of the particle as $t \to \infty$?

Does the non-relativistic limit of your expression have the expected form?

2. A charge and a wire

A point charge, Q, moves at a speed $V \ll c$ parallel to a wire carrying a current I and zero net charge density. The conduction charges in the wire move with a drift velocity $v \ll c$.

Show that in the rest frame of the point charge, the wire appears to have a uniform charge density that is $\frac{vV}{c^2}$ times the charge density of conduction charges in the wire (as measured in the wire's rest frame).

Show that the force on Q due to this charge density is the same as the magnetic force on Q measured in the wire's rest frame.

3. Parallel plate capacitor in motion

In its rest frame a parallel plate capacitor has a uniform electric field $\mathbf{E}_0$ between its plates. In the lab frame this capacitor moves at a constant velocity $\mathbf{v}$ in a direction parallel to its plates.

Using the Lorentz transformations, find expressions for the lab-frame electric and magnetic fields, Lorenz-gauge scalar and vector potentials, and the charge and current densities on the plates.

Check that these lab-frame quantities are consistent with each other (e.g. that the electric and magnetic fields are generated by the charges and currents you find).

4. Covariant field angular momentum and conservation laws

Consider the rank-three Lorentz tensor, $M^{\alpha\beta\gamma} \equiv \Theta^{\alpha\beta}x^\gamma - \Theta^{\alpha\gamma}x^\beta$, constructed out of the position four-vector and the symmetric stress-energy tensor.

(a) Making use of properties of $\Theta^{\alpha\beta}$ that we've already proven, show that $\partial_\alpha M^{\alpha\beta\gamma} = 0$.

(b) Suppose that source-free electromagnetic fields exist in a localized region of space. Consider the integral of $\partial_\alpha M^{\alpha\beta\gamma}$ over all space.

(i) Show that if β and γ are both spatial indices, the following quantity is conserved: $\int d^3\mathbf{x} \left(\Theta^{0i}x^j - \Theta^{0j}x^i\right)$. Show that this quantity can be written in terms of the momentum density of the fields and interpret it in terms of the angular momentum of the fields.

(ii) Show that when $\beta = 0$ we have a conservation law, $\frac{d\mathbf{X}}{dt} = \frac{\mathbf{P}}{U/c^2}$, where $\mathbf{P}$ is the total momentum of the fields, U is their total energy and $\mathbf{X}$ is the "center-of-energy" position defined by $\mathbf{X}\int d^3\mathbf{x}\, u = \int d^3\mathbf{x}\,\mathbf{x}\, u$. Compare this with the center-of-mass of a system of point particles (or a continuous distribution of mass).

5. A moving current loop

A small current loop having a magnetic moment $\mathbf{m}$ in its own rest frame is in motion with a constant velocity, $\mathbf{v}$, as measured in the lab frame. In its rest frame, the loop causes a magnetic vector potential $\frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}'}{r'^3}$ and zero scalar potential. Show that in the lab frame the vector and scalar potentials are given by

$$\mathbf{A}(\mathbf{r},t) = \gamma\frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{R}}{\left(\gamma^2\,\mathbf{R}_\parallel^2 + \mathbf{R}_\perp^2\right)^{3/2}}, \qquad \varphi(\mathbf{r},t) = \gamma\frac{\mu_0}{4\pi}\frac{\mathbf{v}\cdot\mathbf{m}\times\mathbf{R}}{\left(\gamma^2\,\mathbf{R}_\parallel^2 + \mathbf{R}_\perp^2\right)^{3/2}},$$

where $\mathbf{R} = \mathbf{r} - \mathbf{v}t$, and $\mathbf{r}, t$ are the lab-frame position and time. The parallel and perpendicular symbols refer to the direction of $\mathbf{v}$.

[*Hint:* separate $\mathbf{A}'$ into components perpendicular and parallel to $\mathbf{v}$ since these components transform differently under the required boost.]

Consider the case where the loop is moving at a non-relativistic speed, $v \ll c$, and show that the moving loop produces a field compatible with it having both a magnetic dipole moment and an electric dipole moment, and compute the magnitude of that electric dipole moment.

6. A moving point charge

A charge q moving along the z-axis of the lab at a constant speed $v = \beta c$ produces an electric field

$$\mathbf{E} = \hat{\mathbf{r}}\frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\frac{1-\beta^2}{\left(1-\beta^2\sin^2\theta\right)^{3/2}}, \tag{1}$$

where $\mathbf{r}$ is the position in the lab relative to the position of the charge at time t.

Show that, as it must, this field satisfies Gauss's law in the lab: $\int d\mathbf{S}.\mathbf{E} = q/\epsilon_0$, where you may choose the integration surface surrounding the charge to make the calculation as simple as possible.

7. A minimally radiating charge

(a) An object moving in one dimension follows a path $x(t)$ starting at rest at $t = 0$ at position $x(0) = 0$ and coming to rest again at $t = T$ at position $x(T) = d$. Show that an arbitrary functional of the object's acceleration,

$$I = \int_0^T dt\, G[\ddot{x}(t)], \tag{2}$$

takes its extremal value when $\frac{d^2}{dt^2}\left(\frac{dG}{d\ddot{x}}\right) = 0$.

(b) Given that a charged particle moving non-relativistically radiates energy at a rate

$$\frac{2}{3}\frac{q^2}{4\pi\epsilon_0}\frac{1}{c^3}\left(\ddot{x}(t)\right)^2, \tag{3}$$

find a formula for the acceleration as a function of time which minimizes the radiated power for a particle following the path described in part (a).