LAGRANGIANS & HAMILTONIANS IN ELECTROMAGNETISM

Before considering the dynamics of charged particles in electromagnetic fields, and the dynamics of the fields themselves, let's refresh our memories on the concept of the Lagrangian and Hamilton's principle:

Suppose a closed mechanical system is described by a set of (generalised) co-ordinates, $q_i(t)$, then the dynamics of the system can be described by a function of these and the corresponding (generalised) velocities, $\dot{q}_i(t)$, known as the Lagrangian:

$$L\left[q_i(t), \dot{q}_i(t)\right].$$

The equations of motion of the system follow from Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad . \quad [L]$$

A simple example is a (non-relativistic) particle moving in a one dimensional potential, where

$$L = T - V = \frac{1}{2} m \dot{x}^2 - V(x)$$

then eqn [L] gives $\quad m\ddot{x} = -\frac{dV}{dx} \quad$ which we recognise as $F = ma$.

We can derive [L] from Hamilton's principle, or "the principle of least action":

We define the action for a trajectory between times t_1 and t_2 as

$$S\left[q_i(t)\right] = \int_{t_1}^{t_2} dt \, L\left[q_i(t), \dot{q}_i(t)\right]$$

and try to find the trajectory choice which extremizes (typically the minimum is the physically relevant extremum) the action.

Suppose we consider variations away from the minimizing trajectory, $\eta_i(t)$,

$q_i(t) = \eta_i(t) + \delta q_i(t)$

with $\delta q_i(t_1) = \delta q_i(t_2) = 0$ so we always start and end in the correct place,

then the variation in the action is

$$\delta S = S[q_i] - S[\eta_i] = \int_{t_1}^{t_2} dt \left[L[\eta_i + \delta q_i, \dot{\eta}_i + \delta \dot{q}_i] - L[\eta_i, \dot{\eta}_i] \right]$$

$$= \int_{t_1}^{t_2} dt \left[\left.\frac{\partial L}{\partial q_i}\right|_{\eta_i} \delta q_i + \left.\frac{\partial L}{\partial \dot{q}_i}\right|_{\eta_i} \delta \dot{q}_i + O(\delta q_i^{(\cdot)})^2 \right]$$

and $\delta \dot{q}_i = \frac{d}{dt} \delta q_i$

$$\delta S = \int_{t_1}^{t_2} dt \left[\left.\frac{\partial L}{\partial q_i}\right|_{\eta_i} \delta q_i - \frac{d}{dt}\left(\left.\frac{\partial L}{\partial \dot{q}_i}\right|_{\eta_i} \right) \delta q_i \right] + \underbrace{\left[\left.\frac{\partial L}{\partial \dot{q}_i}\right|_{\eta_i} \delta q_i \right]_{t_1}^{t_2}}_{\to 0}$$

after integrating by parts

$$\Rightarrow \delta S = \int_{t_1}^{t_2} dt\, \delta q_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) \right]_{\eta_i} = 0 \leftarrow \text{to be the extremum}$$

and thus, since δq_i is completely arbitrary, it must be that the object in square brackets, evaluated on the extremizing path is zero, which gives us the Lagrange equations, [L].

There is not a unique choice of Lagrangian for any given system, since we can always add a total time derivative and not change [L]:

$$L \to L + \frac{d}{dt} X[q_i, \dot{q}_i] \qquad \text{(action only changed by an irrelevant constant)}$$

Another useful quantity is the Hamiltonian,

$$H(q_i, p_i) \equiv p_i \dot{q}_i - L[q_i, \dot{q}_i]$$

which features the CANONICAL MOMENTA : $p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$

If we allow the Lagrangian to have explicit dependence on time (e.g. due to some external time varying force), $L[q_i, \dot{q}_i, t]$, then

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i$$

The Lagrange equations still hold so $\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{d}{dt} p_i = \dot{p}_i$ $\quad [L_p]$

and $\frac{dL}{dt} = \frac{\partial L}{\partial t} + \dot{p}_i \dot{q}_i + p_i \ddot{q}_i = \frac{\partial L}{\partial t} + \frac{d}{dt}(p_i \dot{q}_i)$

$$\frac{\partial L}{\partial t} = \frac{d}{dt}\left(L - p_i \dot{q}_i\right) = -\frac{dH}{dt}$$

and so if the Lagrangian is not explicitly time dependent $\frac{dH}{dt} = 0$ & the Hamiltonian is a conserved quantity.

We may also derive an alternative expression for the equations of motion, known as Hamilton's equations :

$$H = p_i \dot{q}_i - L[q_i, \dot{q}_i] \rightarrow dH = p_i d\dot{q}_i + \dot{q}_i dp_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i$$

$$= \underbrace{\left(p_i - \frac{\partial L}{\partial \dot{q}_i}\right)}_{=0 \text{ by definition of } p_i} d\dot{q}_i + \dot{q}_i dp_i - \dot{p}_i dq_i \quad (\text{using } [L_p])$$

$$dH = \dot{q}_i dp_i - \dot{p}_i dq_i = \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i$$

$$\Rightarrow \boxed{\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}}$$ Hamilton's equations

the Lagrangian for a free relativistic particle

The action, $S = \int_{t_1}^{t_2} dt\, L$, in the relativistic setting is more naturally expressed in terms of the proper time, τ, ie. the time measured in the particle's rest frame. We earlier showed that $dt = \gamma\, d\tau$ so

$$S = \int_{\tau_1}^{\tau_2} d\tau\, \gamma L ,$$

and since the action should be a Lorentz scalar (to ensure the equations of motion take the same form in all inertial frames), we conclude that γL should also be a Lorentz scalar.

Now for a free particle, the lagrangian cannot be a function of its position, but only of its velocity. Thus we seek a Lorentz invariant featuring the velocity. The obvious four-vector containing the velocity is

$$U^\mu = \gamma(u)\left[c, \vec{u}\right]$$

but the only Lorentz scalar we can form from this is $U_\mu U^\mu$ which equals c^2 and is thus independent of the velocity.

It follows that $\gamma L = fn(U_\mu U^\mu) = fn(c^2) = \text{constant}$

$$\Rightarrow L = \frac{\text{const}}{\gamma} = \alpha\sqrt{1 - u^2/c^2}$$

then the Lagrange equation $\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{u}}\right) = 0 \longrightarrow 0 = \frac{d}{dt}\left(\alpha \left(\frac{-2\vec{u}}{c^2}\right)\frac{1/2}{\sqrt{1-u^2/c^2}}\right)$

or $0 = \frac{d}{dt}\left(-\frac{\alpha}{c^2}\gamma\vec{u}\right)$ where we identify $-\frac{\alpha}{c^2}\gamma\vec{u}$ as the relativistic momentum, $\vec{p} = \gamma m \vec{u}$

and thus set $\alpha = -mc^2$

$$\boxed{L_{free} = -mc^2\sqrt{1 - u^2/c^2} = \frac{-mc^2}{\gamma(u)}}$$ (free particle of mass m)

N.B. for $u \ll c$ $\quad L_{free} \rightarrow -mc^2\left(1 - \frac{u^2}{2c^2} + \dots\right) = -mc^2 + \frac{1}{2}mu^2 + \dots$

Notice that $S = -mc^2\int_{\tau_1}^{\tau_2} d\tau$ so the path of minimum action is the path of largest proper time.

the Lagrangian for a charged particle in an external e/m field

We can infer the form this must take by considering the non-relativistic limit, $u \ll c$. In that case we expect $L = T - V$, and $V = q\varphi$ with φ the scalar potential.

Now φ isn't a Lorentz invariant, but γL should be a Lorentz scalar. The relevant four-vector containing φ is $A^\mu = [\varphi/c, \vec{A}]$.

We can construct Lorentz scalars $x_\mu A^\mu$ and $U_\mu A^\mu$ but the first of these we reject in order to have a translationally invariant action.

A Lagrangian having the right non-relativistic limit is

$$L = L_{free} - \frac{1}{\gamma} q U_\mu A^\mu = L_{free} - q\varphi + q\vec{u}\cdot\vec{A}$$

and we can check that this gives the expected equation of motion by evaluating the Lagrange eqn

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{u}}\right) = \frac{\partial L}{\partial \vec{r}} \quad [L]$$

$$\frac{\partial L}{\partial \vec{u}} = \underbrace{\gamma m \vec{u}}_{\vec{p}} + q\vec{A} \quad \Bigg| \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \vec{u}}\right) = \frac{d\vec{p}}{dt} + q\frac{d}{dt}\vec{A}(\vec{r},t) = \frac{d\vec{p}}{dt} + q\left(\frac{\partial \vec{A}}{\partial t} + \frac{\partial \vec{r}}{\partial t}\cdot\vec{\nabla}\vec{A}\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{u}}\right) = \frac{d\vec{p}}{dt} + q\frac{\partial \vec{A}}{\partial t} + q\vec{u}\cdot\vec{\nabla}\vec{A}$$

$$\frac{\partial L}{\partial \vec{r}} = -q\vec{\nabla}\varphi + q\vec{\nabla}(\vec{u}\cdot\vec{A}) = -q\vec{\nabla}\varphi + q\vec{u}\cdot\vec{\nabla}A + q\vec{u}\times(\vec{\nabla}\times\vec{A})$$

$$[L] \rightarrow \quad \frac{d\vec{p}}{dt} + q\frac{\partial \vec{A}}{\partial t} + \cancel{q\vec{u}\cdot\vec{\nabla}\vec{A}} = -q\vec{\nabla}\varphi + q\vec{u}\times(\vec{\nabla}\times\vec{A}) + \cancel{q\vec{u}\cdot\vec{\nabla}\vec{A}}$$

$$\frac{d\vec{p}}{dt} = q\left(-\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}\right) + q\vec{u}\times(\vec{\nabla}\times\vec{A})$$

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \vec{u}\times\vec{B}\right).$$

the canonical momentum of the particle in the external field can be obtained

$$\vec{P} \equiv \frac{\partial L}{\partial \vec{u}} = \gamma m \vec{u} + q\vec{A} = \vec{p} + q\vec{A}.$$

It follows that $\gamma \vec{u} = \frac{1}{m}(\vec{P} - q\vec{A})$

so $\gamma^2 u^2 = \frac{1}{m^2}(\vec{P}-q\vec{A})\cdot(\vec{P}-q\vec{A}) = X$

$$\frac{u^2}{1-u^2/c^2} = X \quad \Rightarrow \quad u^2 = X - X/c^2\, u^2$$

$$u^2(1 + X/c^2) = X \qquad u^2 = X/1+X/c^2$$

$$1 - u^2/c^2 = \frac{1+X/c^2}{1+X/c^2} - \frac{X/c^2}{1+X/c^2} = \frac{1}{1+X/c^2}$$

$$\gamma = \frac{1}{\sqrt{1-u^2/c^2}} = \sqrt{1+X/c^2}$$

and $\vec{u} = \frac{1}{m}\frac{1}{\sqrt{1+\frac{1}{m^2c^2}(\vec{P}-q\vec{A})^2}}(\vec{P}-q\vec{A}) = \frac{c}{\sqrt{m^2c^2+(\vec{P}-q\vec{A})^2}}(\vec{P}-q\vec{A})$

So the Hamiltonian $H = \vec{u}\cdot\vec{P} - L$ can we written

$$H = \vec{u}\cdot\vec{P} - \left(-\frac{mc^2}{\gamma} - q\varphi + q\vec{u}\cdot\vec{A}\right) = \frac{mc^2}{\gamma} + q\varphi + \vec{u}\cdot(\vec{P}-q\vec{A})$$

$$= q\varphi + \frac{mc^2}{\sqrt{1+(\vec{P}-q\vec{A})^2/m^2c^2}} + \frac{c}{\sqrt{m^2c^2+(\vec{P}-q\vec{A})^2}}(\vec{P}-q\vec{A})^2$$

$$H = q\varphi + \frac{m^2c^3 + c(\vec{P}-q\vec{A})^2}{\sqrt{m^2c^2+(\vec{P}-q\vec{A})^2}} = \underline{c\sqrt{m^2c^2+(\vec{P}-q\vec{A})^2} + q\varphi}.$$

Notice that if we consider the four-momentum $p^\mu = [E/c, \vec{p}]$ and use $\vec{p} = \vec{P} - q\vec{A}$ and assume that $E = H - q\varphi$, we have

$$p_\mu p^\mu = m^2c^2 = (H-q\varphi)^2/c^2 - (\vec{P}-q\vec{A})^2$$

$$\Rightarrow H = c\sqrt{m^2c^2+(\vec{P}-q\vec{A})^2} + q\varphi$$

and we see that the Hamiltonian appears in the time component of a four-vector.

the lagrangian for the electromagnetic field

In order to find this, we need to generalize our concept of a dynamical system from one of a discrete set of co-ordinates, $q_i(t)$ with $i=1,2,3...$ to coordinates that vary continuously with position, $q(\vec{r},t)$.

In doing so, if we want to end up with something that can be Lorentz covariant, as well as derivatives with respect to time, as in $\dot{q}_i(t)$, we'll also need derivatives with respect to spatial position.

The appropriate correspondence is

$$i, t \longrightarrow x^\mu$$
$$q_i(t) \longrightarrow \varphi(x)$$
$$\dot{q}_i(t) \longrightarrow \partial^\mu \varphi(x)$$

$$L = \sum_i L_i(q_i, \dot{q}_i) \longrightarrow \int d^3x\, \mathcal{L}(\varphi, \partial^\mu \varphi)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \longrightarrow \partial^\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi)}\right) - \frac{\partial \mathcal{L}}{\partial \varphi}$$ "Euler-Lagrange equations" [EL]

$$S = \int dt\, L \longrightarrow S = \int d^4x\, \mathcal{L}$$

where $\varphi(x)$ is a FIELD on a continuous spacetime. If we want to describe multiple fields then we can add an extra discrete index, $\varphi_k(x)$.

The invariance of d^4x under Lorentz transformations indicates that the LAGRANGIAN DENSITY, $\mathcal{L}$, should be Lorentz invariant.

For electromagnetism, appropriate field "co-ordinates" would seem to be $\varphi_k \to A^\alpha$ with the "velocities" being $\partial^\beta A^\alpha$.

A Lorentz-invariant form, quadratic in the fields A^α is $F_{\mu\nu}F^{\mu\nu}$, so we might guess that the "free-field" part of the Lagrangian is proportional to this.

The invariant quantity $J_\mu A^\mu = \rho\varphi - \vec{J}.\vec{A}$ looks like an interaction energy between fields and sources, so it may appear too.

Our guess is that $\mathcal{L}_{em} = -\frac{1}{4\mu_0} F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu$

and we can easily check that this $\mathcal{L}$ in the [EL] eqns gives us Maxwell's eqns

$$\mathcal{L}_{em} = \frac{-1}{4\mu_0} F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\mathcal{L}_{em} = -\frac{1}{4\mu_0} g_{\mu\alpha} g_{\nu\beta} \left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) - J_\mu A^\mu$$

$$\frac{\partial \mathcal{L}}{\partial(\partial^\rho A^\sigma)} = -\frac{1}{4\mu_0} g_{\mu\alpha} g_{\nu\beta} \left[\left(\delta_\rho{}^\alpha \delta_\sigma{}^\beta - \delta_\rho{}^\beta \delta_\sigma{}^\alpha\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) + \left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right)\left(\delta_\rho{}^\mu \delta_\sigma{}^\nu - \delta_\rho{}^\nu \delta_\sigma{}^\mu\right)\right]$$

$$= -\frac{1}{4\mu_0} \left[\left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) + \left(g_{\rho\alpha} g_{\sigma\beta} - g_{\sigma\alpha} g_{\rho\beta}\right)\left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right)\right]$$

$$= -\frac{1}{4\mu_0} \left[\partial_\rho A_\sigma - \partial_\sigma A_\rho - \partial_\sigma A_\rho + \partial_\rho A_\sigma + \partial_\rho A_\sigma - \partial_\sigma A_\rho - \partial_\sigma A_\rho + \partial_\rho A_\sigma\right]$$

$$= -\frac{1}{4\mu_0} \left[4\partial_\rho A_\sigma - 4\partial_\sigma A_\rho\right] = \underline{-\frac{1}{\mu_0} F_{\rho\sigma}}$$

$$\underline{\frac{\partial \mathcal{L}}{\partial A^\sigma} = -J_\sigma}$$

$\Rightarrow$ [EL] $\longrightarrow$ $-\frac{1}{\mu_0}\partial^\rho\left(F_{\rho\sigma}\right) - \left(-J_\sigma\right) = 0 \quad \rightsquigarrow \quad \boxed{\partial^\rho F_{\rho\sigma} = \mu_0 J_\sigma}$ [M] ✓ (see page 106)

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auge invariance?

since [M] is gauge invariant, it follows that the condition from which it may be derived, $\delta S = 0$, remains true under changes of gauge

considering the interaction part $S = -\int d^4x\, J_\mu A^\mu$ we can make a gauge transformation $A^\mu \to A^\mu + \partial^\mu(\delta\Lambda)$ then

$$\delta S = 0 = \int d^4x\, J_\mu \partial^\mu(\delta\Lambda) = \int d^4x\, \partial^\mu\left[J_\mu \delta\Lambda\right] - \int d^4x\, \delta\Lambda\, \partial^\mu J_\mu$$

but the first integral can be transformed into a surface integral at set equal to zero for localized currents, leaving

$$0 = \int d^4x\, (\delta\Lambda)(\partial^\mu J_\mu)$$

and since $\delta\Lambda(x)$ can take arbitrary form $\Rightarrow$ $0 = \partial^\mu J_\mu$, which is the conservation of charge.

$\Rightarrow$ (local) conservation of charge $\longleftrightarrow$ (local) gauge invariance of the theory

Noether's theorem : symmetries & conserved quantities

Let's consider a more general action : $S = \int d^4x \, \mathcal{L}\left[x^\mu, A^\alpha(x^\mu), \partial^\beta A^\alpha(x^\mu)\right]$

Suppose we make an infinitesimal transformation of the co-ordinates : $x^\mu \rightarrow x'^\mu = x^\mu + \delta x^\mu$

The field transformation will be $A^\alpha(x) \rightarrow A'^\alpha(x') = A^\alpha(x) + \delta A^\alpha(x)$

where $\delta A^\alpha(x)$ accounts for both the change in the co-ordinates & the change in the fields

$A'(x')$ = transformed field at transformed position

We can also consider $\underbrace{A'^\alpha(x)}_{\text{transformed field at the original position}} = A^\alpha(x) + \bar{\delta} A^\alpha(x)$

$\bar{\delta}A$ and δA are related :

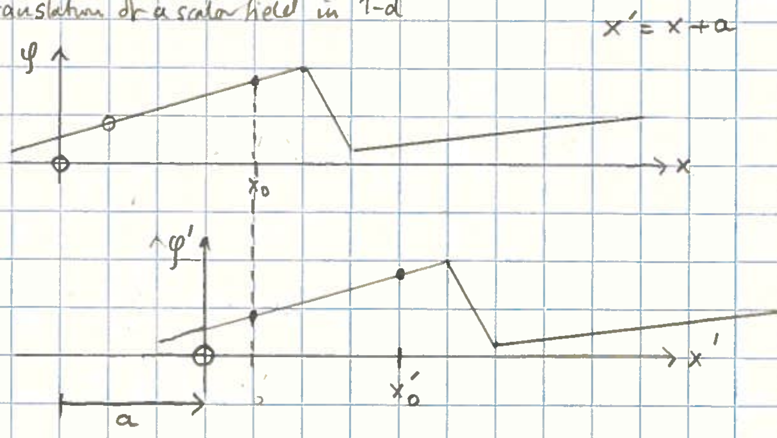
$$A'^\alpha(x') = A'^\alpha(x + \delta x) = A'^\alpha(x) + \frac{\partial A'^\alpha}{\partial x^\mu}\delta x^\mu + \dots$$

$$= A^\alpha(x) + \bar{\delta} A^\alpha(x) + \frac{\partial A^\alpha}{\partial x^\mu}\delta x^\mu + \dots \quad \left(\text{to first order}\right)$$

$$\Rightarrow \underline{\delta A^\alpha = \bar{\delta} A^\alpha + \frac{\partial A^\alpha}{\partial x^\mu}\delta x^\mu}$$

if we want to make a transformation of a field, without changing co-ordinates we can do this with $\delta x^\mu = 0$ but $\bar{\delta} A^\alpha \neq 0$.

A simple transformation : translation of a scalar field in 1-d



$\varphi'(x') = \varphi(x) \quad \Rightarrow \delta\varphi = 0$ in this case

$\varphi'(x) = \varphi(x - a) \quad \Rightarrow \bar{\delta}\varphi = \varphi(x-a) - \varphi(x)$

The derivation of Noether's theorem involves some potentially confusing steps – let's make it as simple as possible by working in one dimension with a scalar field

$$x \to x' = x + \delta x$$
$$\varphi(x) \to \varphi'(x') = \varphi(x) + \delta\varphi(x)$$
$$\varphi'(x) = \varphi(x) + \bar{\delta}\varphi(x)$$

$$\delta\varphi = \bar{\delta}\varphi + \frac{d\varphi}{dx}dx$$

$$S = \int_a^b dx\, \mathcal{L}\left(x, \varphi(x), \frac{d\varphi}{dx}(x)\right) \longrightarrow S' = \int_{a+\delta a}^{b+\delta b} dx'\, \mathcal{L}\left(x', \varphi'(x'), \frac{d\varphi'}{dx'}(x')\right)$$

but x' is just a dummy variable, so we can relabel it x

$$S' = \int_{a+\delta a}^{b+\delta b} dx\, \mathcal{L}\left(x, \varphi'(x), \frac{d\varphi'}{dx}(x)\right)$$

hence $$\delta S = \int_{a+\delta a}^{b+\delta b} dx\, \mathcal{L}\left(x, \varphi'(x), \frac{d\varphi'}{dx}(x)\right) - \int_a^b dx\, \mathcal{L}\left(x, \varphi(x), \frac{d\varphi}{dx}(x)\right)$$

and to first order in infinitesimals

$$\delta S = \underbrace{\int_{a+\delta a}^{b+\delta b} dx\, \mathcal{L}\left(x, \varphi, \frac{d\varphi}{dx}\right) - \int_a^b dx\, \mathcal{L}\left(x, \varphi, \frac{d\varphi}{dx}\right)} + \int_a^b dx \frac{\partial \mathcal{L}}{\partial \varphi}\bar{\delta}\varphi + \int_a^b dx \frac{\partial \mathcal{L}}{\partial\left(\frac{d\varphi}{dx}\right)}\bar{\delta}\left(\frac{d\varphi}{dx}\right)$$

$$\int_b^{b+\delta b} dx\, \mathcal{L} - \int_a^{a+\delta a} dx\, \mathcal{L} \longrightarrow \delta b \cdot \mathcal{L}(b) - \delta a \cdot \mathcal{L}(a) = \int_a^b dx \frac{d}{dx}\left[\delta x \mathcal{L}\right]$$

$$\Rightarrow \delta S = \int_a^b dx \left[\frac{d}{dx}(\delta x \mathcal{L}) + \bar{\delta}\varphi \frac{\partial \mathcal{L}}{\partial \varphi} + \bar{\delta}\left(\frac{d\varphi}{dx}\right)\frac{\partial \mathcal{L}}{\partial\left(\frac{d\varphi}{dx}\right)}\right]$$

using the Euler-Lagrange eqn $\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{d\varphi}{dx}\right)}\right)$

and since $\bar{\delta}\varphi$ is evaluated at a fixed position, $\bar{\delta}\left(\frac{d\varphi}{dx}\right) = \frac{d}{dx}\bar{\delta}\varphi$

$$\Rightarrow \delta S = \int_a^b dx \frac{d}{dx}\left[\delta x \mathcal{L} + \bar{\delta}\varphi \frac{\partial \mathcal{L}}{\partial\left(\frac{d\varphi}{dx}\right)}\right]$$

$$= \int_a^b dx \frac{d}{dx}\left[\delta x\left(\mathcal{L} - \frac{d\varphi}{dx}\frac{\partial \mathcal{L}}{\partial\left(\frac{d\varphi}{dx}\right)}\right) + \delta\varphi \frac{\partial \mathcal{L}}{\partial\left(\frac{d\varphi}{dx}\right)}\right]$$

and if the transformation is a symmetry of the system, $\delta S = 0$.

the 3+1 dimensional analogue for four-vector fields is

$$0 = \partial_\mu \left[\left(\mathcal{L} g^{\mu\nu} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\alpha)} \partial^\nu A^\alpha \right) \delta x_\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\alpha)} \delta A^\alpha \right]$$

the implication is that the quantity in square brackets gives us a conserved quantity when the transformation leaves the action invariant.

e.g. UNIFORM TRANSLATION IN SPACE-TIME: $\delta x_\mu = \varepsilon_\mu$ (constant infinitesimal four-vector), $\delta A^\alpha = 0$

$$\Rightarrow \partial_\mu \left[\mathcal{L} g^{\mu\nu} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\alpha)} \partial^\nu A^\alpha \right] = 0$$

without sources:

$$\mathcal{L}_{em} = -\frac{1}{4\mu_0} F_{\rho\sigma} F^{\rho\sigma} \qquad F_{\rho\sigma} = \partial_\rho A_\sigma - \partial_\sigma A_\rho$$

$$\frac{\partial \mathcal{L}_{em}}{\partial(\partial_\mu A^\alpha)} = -\frac{1}{\mu_0} F^\mu{}_\alpha \qquad \text{(see page 130)}$$

$$-\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\alpha)} \partial^\nu A^\alpha \right] = \frac{1}{\mu_0} \partial_\mu \left[F^\mu{}_\alpha \left(F^{\nu\alpha} + \partial^\alpha A^\nu \right) \right]$$

$$= \frac{1}{\mu_0} \left[\partial_\mu \left(F^\mu{}_\alpha F^{\nu\alpha} \right) + \partial^\alpha A^\nu \underbrace{\partial_\mu F^\mu{}_\alpha}_{\substack{=0 \\ \text{sourceless}}} + \underbrace{F^\mu{}_\alpha \partial_\mu \partial^\alpha A}_{\substack{=0 \\ \text{antisym} \\ \times \text{sym}}} \right]$$

$$= \frac{1}{\mu_0} \partial_\mu \left(F^\mu{}_\alpha F^{\nu\alpha} \right)$$

so $$0 = \partial_\mu \frac{1}{\mu_0} \left[F^\mu{}_\alpha F^{\nu\alpha} - \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right] = \partial_\mu \frac{1}{\mu_0} \left[F^\mu{}_\alpha F^{\alpha\nu} + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right]$$

and we recognise $\Theta^{\mu\nu} = \frac{1}{\mu_0} \left[F^\mu{}_\alpha F^{\alpha\nu} + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right]$ as the "stress-energy tensor" (pgs 109/110)

$$\partial_\mu \Theta^{\mu\nu} = 0 .$$

We come to an observation of fundamental importance – the conservation of momentum and energy in free electromagnetic fields can be traced back to the translational invariance of the action of the theory.