MATH PRELIMINARIES

The wave equation: when we come to consider Maxwell's equations in generality we'll find field satisfy an equation of the form

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\psi(\vec{r},t) = f(\vec{r},t).$$

First up lets consider the one-dimensional version of this equation without a source term ($f=0$)

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\psi(x,t) = 0$$

which can be easily solved by changing variables to $\xi = x+ct$, $\eta = x-ct$ since then

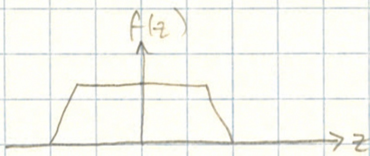
$$\left.\frac{\partial}{\partial x}\right)_t = \left.\frac{\partial \xi}{\partial x}\right)_t \frac{\partial}{\partial \xi} + \left.\frac{\partial \eta}{\partial x}\right)_t \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$

& similarly $\left.\frac{1}{c}\frac{\partial}{\partial t}\right)_x = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}$ so that $\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = 4\frac{\partial^2}{\partial \xi \partial \eta}$

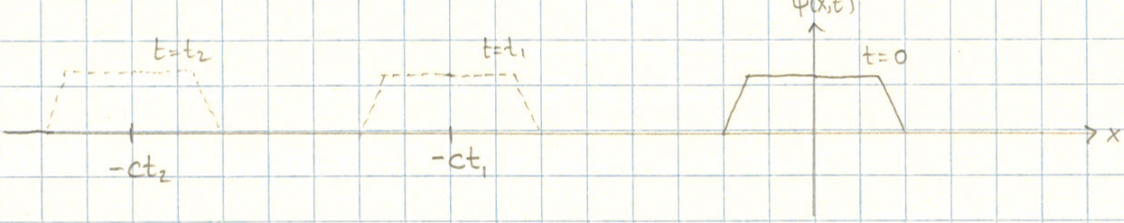
and thus $\frac{\partial^2 \psi(\xi,\eta)}{\partial \xi \partial \eta} = 0$ which is solved by $\psi = f(\xi) + g(\eta)$

or $\underline{\psi(x,t) = f(x+ct) + g(x-ct)}$ where f, g are any well-behaved functions.

e.g. suppose $f(z)$ is a localized function ($\& \, g(z)=0$)



then the time-evolution of $\psi(x,t)$ will be



& the "bump" is observed to move left at a speed c

Similarly $g(x-ct)$ corresponds to disturbances moving to the right at speed c.

Generalizing to three space dimensions, $f(x \pm ct)$ is still a solution to $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi(\vec{r},t) = 0$ – it's a particular solution called a "<u>plane-wave</u>" since every value of y, z has the same amplitude.

Returning to the origin problem, in three-dimensions, with a source,

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\psi(\vec{r},t) = f(\vec{r},t),$$

we can seek a general solution to this equation using Green's functions:

if we can solve $\left[\nabla_r^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] G(\vec{r}t;\vec{r}'t') = -4\pi\,\delta^{(3)}(\vec{r}-\vec{r}')\,\delta(t-t')$ [G]

for $G(\vec{r}t;\vec{r}'t')$ we can construct a solution as

$$\psi(\vec{r},t) = -\frac{1}{4\pi}\int d^3\vec{r}'dt'\,G(\vec{r}t;\vec{r}'t')\,f(\vec{r}',t') + \varphi(\vec{r},t) \quad [S]$$

[assuming only boundaries at infinity]

[for ... m ... ies]

where the solutions to the homogeneous eqn $\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\varphi = 0$

will be explored along the way.

Our task then is to solve [G] — first we note that we can change variables to $\vec{\rho} = \vec{r}-\vec{r}'$, $\tau = t-t'$ & that $G(\vec{\rho},\tau)$ can only depend upon $|\vec{\rho}|$

$$\left[\nabla_\rho^2 - \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2}\right] G(\rho,\tau) = -4\pi\,\delta(\vec{\rho})\,\delta(\tau)$$

$$\& \quad \nabla^2 G = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial G}{\partial\rho}\right) = \frac{2}{\rho}\frac{\partial G}{\partial\rho} + \frac{\partial^2 G}{\partial\rho^2} = \frac{1}{\rho}\frac{\partial^2}{\partial\rho^2}(\rho G)$$

$$\text{so} \quad \frac{1}{\rho}\underbrace{\left[\frac{\partial^2}{\partial\rho^2} - \frac{1}{c^2}\frac{\partial^2}{\partial\tau^2}\right](\rho G)}_{\text{this is the one-dim wave eqn we already studied}} = -4\pi\,\delta(\vec{\rho})\,\delta(\tau)$$

so, except at $\vec{\rho} = \vec{0}$, we have $\rho G(\rho,\tau) = g_\pm(\tau \pm \rho/c)$ for any functions $g_\pm(s)$

$$\text{now} \quad \nabla^2\left(\frac{g_\pm}{\rho}\right) = \vec{\nabla}\cdot\left[\frac{1}{\rho}\vec{\nabla}g + g\vec{\nabla}\frac{1}{\rho}\right] = -\frac{\hat{r}}{\rho^2}\cdot\vec{\nabla}g + \frac{1}{\rho}\nabla^2 g - \frac{\hat{r}}{\rho^2}\cdot\vec{\nabla}g + g\,\nabla^2\left(\frac{1}{r}\right)$$

$$= -\frac{2}{\rho^2}\frac{\partial}{\partial\rho}g + \frac{1}{\rho}\left(\frac{2}{\rho}\frac{\partial g}{\partial\rho} + \frac{\partial^2 g}{\partial\rho^2}\right) + g\cdot\left(-4\pi\,\delta^{(3)}(\vec{\rho})\right) = \frac{1}{\rho}\frac{\partial^2 g}{\partial\rho^2} - 4\pi\,\delta(\vec{\rho})\,g$$

but $\frac{\partial g}{\partial\rho} = \frac{\partial s}{\partial\rho}\frac{d}{ds}g(s) = \pm\frac{1}{c}g'$ $\quad$ so $\nabla^2\left(\frac{g_\pm}{\rho}\right) = \frac{1}{c^2}\frac{1}{\rho}g''_\pm - 4\pi\,\delta(\vec{\rho})\,g_\pm$

$s = \tau \pm \rho/c$

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\left[\frac{g_\pm}{\rho}\right] = \cancel{\frac{1}{c^2}\frac{1}{\rho}g''_\pm} - 4\pi\delta(\vec{\rho})g_\pm \cancel{-\frac{1}{c^2}\frac{1}{\rho}g''_\pm}$$

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\left(\frac{g_\pm}{\rho}\right) = -4\pi\delta(\vec{\rho})g_\pm$$ which is to be compared to

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]G(\rho,\tau) = -4\pi\delta(\vec{\rho})\delta(\tau)$$ in the limit $\rho \to 0$

so $$G_\pm(\rho,\tau) = \frac{1}{\rho}\delta(\tau \pm \rho/c)$$

& thus $$G_\pm(\vec{r},t;\vec{r}'t') = \frac{1}{|\vec{r}-\vec{r}'|}\delta\left(t-t' \pm \frac{1}{c}|\vec{r}-\vec{r}'|\right)$$

$\to$ consider [S] with the choice G_-, then

$$\psi(\vec{r},t) = \varphi(\vec{r},t) - \frac{1}{4\pi}\int d^3\vec{r}'\,\frac{1}{|\vec{r}-\vec{r}'|}f\left(\vec{r}', t - \frac{1}{c}|\vec{r}-\vec{r}'|\right) \quad [R]$$

the "retarded" solution.

and we see that the amplitude at $\vec{r}$ at time t is caused by the action of the source at $\vec{r}'$ at an earlier time, $t' = t - \frac{|\vec{r}-\vec{r}'|}{c}$,

i.e. we must account for the finite time for the disturbance to propagate from $\vec{r}'$ to $\vec{r}$.

An interpretation of $\varphi(\vec{r},t)$ comes if we suppose that at some very early time ($t \to -\infty$ in the way we've done this), there was no source, $f=0$, then $\varphi(\vec{r},t)$ is an "incoming wave".

$\to$ the other option, G_+, gives

$$\psi(\vec{r},t) = \varphi(\vec{r},t) - \frac{1}{4\pi}\int d^3\vec{r}'\,\frac{1}{|\vec{r}-\vec{r}'|}f\left(\vec{r}', t + \frac{1}{c}|\vec{r}-\vec{r}'|\right)$$

the "advanced" solution

and we'll have less cause to use this possible solution

A convenient shorthand for [R] in the case of no incoming wave is

$$\psi(\vec{r},t) = -\frac{1}{4\pi}\int d^3\vec{r}'\,\frac{[f(\vec{r}',t')]_{ret}}{|\vec{r}-\vec{r}'|}$$

where $[\;]_{ret}$ means evaluate t' at $t - \frac{1}{c}|\vec{r}-\vec{r}'|$

The homogeneous or "free" wave equation, $\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\psi(\vec{r},t) = 0$

has spherical wave solutions. We can obtain the Helmholtz equation by Fourier analysing in time

$$\psi(\vec{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \psi(\vec{r},\omega)e^{-i\omega t}$$

where for each frequency ω, $(\nabla^2 + k^2)\psi(\vec{r},\omega) = 0$ $\quad [k^2 = \omega^2/c^2]$

Spherically symmetric solutions, $\psi(r,\omega)$, follow from $\nabla^2\psi(r,\omega) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)$

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) = -k^2\psi \rightarrow \frac{d^2}{dr^2}(r\psi) = -k^2(r\psi) \Rightarrow r\psi = e^{\pm ikr}$$

$$\& \ \psi(r,\omega) = \frac{1}{r}e^{\pm ikr}$$

and for each frequency ω we have a spherical wave $\underline{\psi(r,t) \sim \frac{1}{r}e^{\pm ikr - i\omega t}}$.

consider a point of constant phase $\phi = \pm kr - \omega t$

$$\frac{d\phi}{dt} = 0 = \pm k\frac{dr}{dt} - \omega \Rightarrow \frac{dr}{dt} = \pm \omega/k \quad \text{"phase velocity"}$$

$\Rightarrow$ + case = outgoing wave
$\quad$ − case = incoming wave

We can also find solutions to $\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\psi(\vec{r},t)=0$ that are not spherically symmetric. A Fourier analysis in time gives the homogeneous Helmholtz equation

$$(\nabla^2+k^2)\,\psi(\vec{r},\omega)=0 \qquad (k^2=\omega^2/c^2)$$

& we can express ∇^2 in spherical polar coordinates

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

but we already know the eigenfunctions of the operator in square brackets are the $Y_{\ell m}(\theta,\phi)$
So we may separate our solution as

$$\psi(\vec{r},\omega) = \sum_{\ell m} f_{\ell m}(r)\,Y_{\ell m}(\theta,\phi),$$

and the $f_{\ell m}(r)$ functions satisfy the differential equation

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{\ell(\ell+1)}{r^2}+k^2\right]f_{\ell m}(r)=0 \qquad \text{(N.B. independent of } m\text{)}$$

changing to $f_{\ell m}(r)=\frac{1}{\sqrt{r}}u_\ell(r)$ we find $\quad u_\ell''+\frac{1}{r}u_\ell'+\left(k^2-\frac{(\ell+\frac12)^2}{r^2}\right)u_\ell=0$

defining $x=kr$ we have $\quad x^2\frac{d^2u}{dx^2}+x\frac{du}{dx}+(x^2-p^2)\,u=0 \quad$ with $p=\ell+\frac12$

which is Bessel's equation. It follows that

$$f_{\ell m}(r) = A_{\ell m}\frac{1}{r^{1/2}}J_{\ell+1/2}(kr) + B_{\ell m}\frac{1}{r^{1/2}}N_{\ell+1/2}(kr).$$

It's convenient to define a class of special functions called the spherical Bessel & spherical Hankel functions:

$$j_\ell(x)\equiv\left(\frac{\pi}{2x}\right)^{1/2}J_{\ell+1/2}(x) \quad , \quad n_\ell(x)\equiv\left(\frac{\pi}{2x}\right)^{1/2}N_{\ell+1/2}(x)$$

$$h_\ell^{(1,2)}(x)\equiv j_\ell(x)\pm i\,n_\ell(x)$$

e.g. $j_0(x)=\frac{\sin x}{x}$; $n_0(x)=-\frac{\cos x}{x}$; $h_0^{(1)}(x)=\frac{e^{ix}}{ix}$

$j_1(x)=\frac{\sin x}{x^2}-\frac{\cos x}{x}$; $n_1(x)=-\frac{\cos x}{x^2}-\frac{\sin x}{x}$; $h_1^{(1)}(x)=-\frac{e^{ix}}{x}\left(1+\frac{i}{x}\right)$

MONOCHROMATIC WAVES IN 1-D

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi(x,t) = 0 \quad [W] \qquad \psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \, e^{-i\omega t}\, \psi(x,\omega)$$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\omega^2}{c^2}\right)\psi(x,\omega) = 0$$

$\Rightarrow$ a basis of solutions is $\psi(x,t) \sim e^{\pm ikx - i\omega t}$ with $k = \omega/c$

these are monochromatic waves with phase velocity $V_p = \frac{\omega}{k} = c$.

SUPERPOSITION OF MONOCHROMATIC WAVES

We can write a general solution of the wave equation as a superposition of monochromatic waves

$$\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \, A(k)\, e^{ikx - i\omega t}$$

Now in practice we'll find that inside some materials, waves do not propagate according to the wave eqn [W] but rather in a way that can be described by having a relationship between k & ω that is not as simple as $k = \omega/c$.

Allowing a more general form for $k(\omega)$, or equivalently $\omega(k)$ is called DISPERSION. For this first discussion we'll insist that $\omega(k)$ is a REAL function, and since ω can't depend on whether the wave is left-travelling or right-travelling: $\omega(-k) = \omega(k)$.

our general wave is then $\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \, A(k)\, e^{ikx}\, e^{-i\omega(k)t}$

(N.B. we've integrated a 2nd order diff. eqn - there should be two constants - they are $Re(A)$, $Im(A)$ for A complex)

But physical solutions for classical waves are real & we must remember to take the real part of this expression $\phi = Re\,\psi = \frac{1}{2}(\psi + \psi^*)$

to get an expression for $A(k)$ consider $\psi(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, A(k)\, e^{ikx}$

$$\& \quad \frac{\partial\psi}{\partial t}(x,0) = \frac{-i}{2\pi}\int_{-\infty}^{\infty} dk\, A(k)\, e^{ikx}\, \omega(k)$$

$$\& \int_{-\infty}^{\infty} dx\, e^{-ik'x} \phi(x,0) = \frac{1}{2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, A(k) \int_{-\infty}^{\infty} dx\, e^{i(k-k')x} + \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, A^*(k) \int_{-\infty}^{\infty} dx\, e^{-i(k+k')x}\right]$$

$$= \frac{1}{2}\left(A(k') + A^*(-k')\right)$$

$$\int_{-\infty}^{\infty} dx\, e^{-ik'x} \frac{\partial \phi}{\partial t}(x,0) = \frac{1}{2}\left[\frac{-i}{2\pi}\int_{-\infty}^{\infty} dk\, A(k)\,\omega(k) \int_{-\infty}^{\infty} dx\, e^{i(k-k')x} + \frac{(-i)^*}{2\pi}\int_{-\infty}^{\infty} dk\, A^*(k)\,\omega^*(k) \int_{-\infty}^{\infty} dx\, e^{-i(k'+k)}\right]$$

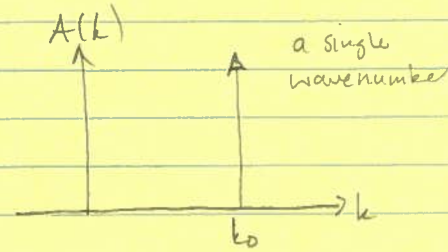
$$= \frac{1}{2}\left[-i\,A(k')\,\omega(k') + i\,A^*(-k')\,\omega(-k')\right] \qquad (\omega \text{ is real})$$

$$= \frac{-i\,\omega(k')}{2}\left[A(k') - A^*(-k')\right]$$

$$\Rightarrow \underline{\int_{-\infty}^{\infty} dx\, e^{-ikx}\left(\phi(x,0) + \frac{i}{\omega(k)}\frac{\partial \phi}{\partial t}(x,0)\right) = A(k)}.$$

For a moment, let's not insist that our solutions are purely real, in which case we have

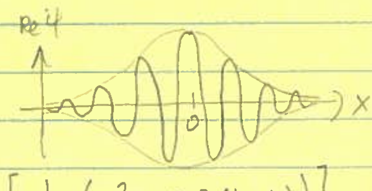
$$A(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx}\, \psi(x,0)$$



and if $\psi(x,0) = e^{ik_0 x}$ then $A(k) = 2\pi\, \delta(k-k_0)$

$$\& \; \psi(x,t) = e^{ikx - i\omega t}$$

But what if we start with a wave that is localized in space, e.g.



$$\psi(x,0) = e^{ik_0 x}\, e^{-x^2/2a^2}$$

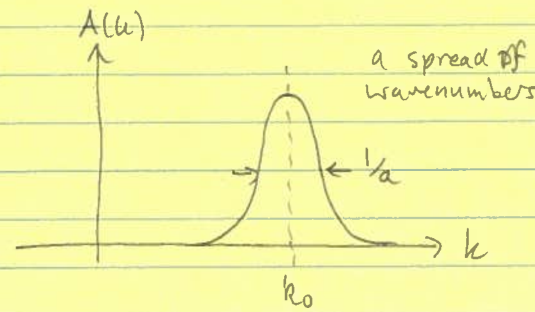
then
$$A(k) = \int_{-\infty}^{\infty} dx\; e^{i(k_0-k)x}\, e^{-x^2/2a^2} = \int_{-\infty}^{\infty} dx\; \exp\left[-\frac{1}{2a^2}\left(x^2 - 2a^2 (k_0-k)x\right)\right]$$

$$= \int_{-\infty}^{\infty} dx\; \exp\left[-\frac{1}{2a^2}\left(x - a^2 i (k_0-k)\right)^2 - \frac{a^2}{2}(k_0-k)^2\right]$$

$$\xi = x/\sqrt{2}a$$

$$= e^{-a^2(k-k_0)^2/2} (\sqrt{2}a) \int_{-\infty}^{\infty} d\xi\; e^{-\xi^2}$$

$$= \sqrt{2\pi}\, a\; e^{-a^2(k-k_0)^2/2}$$



we can examine what happens to $\psi(x,t)$ approximately for any such peaked distribution - expand $\omega(k)$ in a Taylor series around the peaking wavenumber, k_0

$$\omega(k) = \underbrace{\omega(k_0)}_{\omega_0} + \left.\frac{d\omega}{dk}\right|_{k_0} (k-k_0) + \ldots$$

then
$$\psi(x,t) \simeq \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\; A(k)\, e^{ikx}\; e^{-i\left[\omega_0 t + \left.\frac{d\omega}{dk}\right|_0 (k-k_0) t + \ldots\right]}$$

$$= \frac{1}{2\pi}\, e^{-i\left[\omega_0 - \left.\frac{d\omega}{dk}\right|_0 k_0\right] t} \int_{-\infty}^{\infty} dk\; A(k)\; e^{ik\left(x - \left.\frac{d\omega}{dk}\right|_0 t\right)}$$

the integral here can be identified as $u\left(x - \frac{d\omega}{dt}\Big|_0 t, 0\right)$

and thus

$$\psi(x,t) \simeq u\left(x - \frac{d\omega}{dt}\Big|_0 t, 0\right) e^{i\left[\frac{d\omega}{dk}\big|_0 k_0 - \omega_0\right] t}$$

and the original pulse translates undistorted (apart from a changing phase) with a velocity, called the "group velocity"

$$\underline{v_g \equiv \frac{d\omega}{dk}\Big|_{k_0}}$$

this is only a first approximation - in fact the effect of the neglected terms in the Taylor series is to distort by "spreading out" the wavepacket as it propagates.