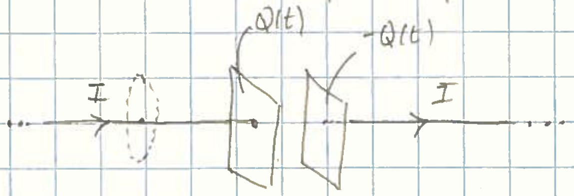
MAXWELL'S EQUATIONS

Last semester we motivated four law's of electromagnetism:

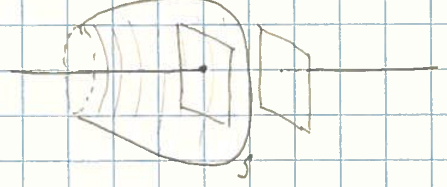
Coulomb's law	$\vec{\nabla}\cdot\vec{D} = \rho$	$\int \vec{D}\cdot d\vec{S} = \int d^3r\, \rho$
Ampère's law (quasistatic: $\vec{\nabla}\cdot\vec{J}=0$)	$\vec{\nabla}\times\vec{H} = \vec{J}$	$\oint_C \vec{H}\cdot d\vec{l} = \int_S d\vec{S}\cdot\vec{J}$
Faraday's law	$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$	$\oint_C \vec{E}\cdot d\vec{l} = -\frac{\partial\Phi}{\partial t}$
Absence of magnetic charges	$\vec{\nabla}\cdot\vec{B} = 0$	$\int \vec{B}\cdot d\vec{S} = 0$

But we can consider a simple experimental setup that indicates that Ampère's law must be incomplete. Consider a long straight wire, carrying current I, that is interrupted by a parallel plate capacitor



consider a circular Ampère loop around the wire, far from the parallel plates — Ampère's law then gives $H = I/2\pi r$ if we choose for the surface S the plane of the circle.

But now suppose we deform S so that it lies between the parallel plates



Now no current density crosses S & $\oint_C \vec{H}\cdot d\vec{l} \neq 0 \Rightarrow H = 0$!?

We may guess the solution by realising that what replaces I in between the plates is a $\vec{D}$-field (allowing a dielectric insertion in generality).
Neglecting end effects, $D = Q/A$ and $\frac{\partial D}{\partial t} = \frac{1}{A}\frac{\partial Q}{\partial t} = I/A$.

So the discrepancy would be removed if Ampère's law were actually

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$$

We can show more generally that our original form of Ampère's law leads to non-conservation of electric charge

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H}) = \vec{\nabla}\cdot\vec{J} + \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{D}$$

$$0 = \vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t}$$

& we see that $\frac{\partial \vec{D}}{\partial t}$ is needed to conserve charge in the general case.

Maxwell's equations:

$$\vec{\nabla}\cdot\vec{D} = \rho \qquad \vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Now we've moved away from static and quasistatic situations, we should reconsider the usefulness of the concept of potentials. We'll temporarily restrict ourselves to the vacuum case where $\vec{D} = \varepsilon_0\vec{E}$ & $\vec{H} = \frac{1}{\mu_0}\vec{B}$ so

$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow$ we can define a magnetic vector potential $\underline{\vec{B} = \vec{\nabla}\times\vec{A}}$

$\vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0} \rightarrow \vec{\nabla}\times\left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0 \Rightarrow$ we can define a scalar potential via

$$\underline{\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}}$$

So using $\varphi, \vec{A}$ ensures that $\vec{\nabla}\cdot\vec{B} = 0$ and Faraday's law are automatically satisfied. The inhomogeneous equations which feature the charge & current sources become

$$\rho = \varepsilon_0\vec{\nabla}\cdot\vec{E} = \varepsilon_0\left(-\nabla^2\varphi - \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A}\right) \Rightarrow \boxed{\nabla^2\varphi + \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = -\rho/\varepsilon_0}$$

$$\mu_0\vec{J} = \vec{\nabla}\times\vec{B} + \varepsilon_0\mu_0\frac{\partial \vec{E}}{\partial t} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) + \varepsilon_0\mu_0\left(-\vec{\nabla}\frac{\partial\varphi}{\partial t} - \frac{\partial^2}{\partial t^2}\vec{A}\right)$$

$$\Rightarrow \boxed{\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t}\right) = -\mu_0\vec{J}}$$

defining $c^2 = \frac{1}{\varepsilon_0\mu_0}$

→ a pair of coupled equations for $\varphi, \vec{A}$

The equations can be decoupled if we take advantage of the GAUGE INVARIANCE of electromagnetism:

notice that $(\varphi, \vec{A})$ and $\left(\varphi - \frac{\partial \Lambda}{\partial t}, \vec{A} + \vec{\nabla}\Lambda\right)$

both describe the same fields $\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

for any choice of the scalar function $\Lambda(\vec{r}, t)$.

Suppose we choose Λ so that the following condition

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0$$ "the Lorenz condition"

is satisfied, then the inhomogeneous eqns for $\varphi, \vec{A}$ decouple:

$$\boxed{\begin{aligned} \left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\varphi &= -\rho/\varepsilon_0 \\ \left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{A} &= -\mu_0 \vec{J} \end{aligned}}$$

"Lorenz gauge"

note that these equations are in the form of the wave eqns we discussed earlier

A different gauge choice, the "Coulomb", "transverse", or "radiation" gauge, does not decouple the equations, but does lead to a particularly simple form for φ. This gauge corresponds to enforcing $\vec{\nabla} \cdot \vec{A} = 0$

so that $\nabla^2 \varphi = -\rho/\varepsilon_0 \Rightarrow \varphi(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0}\int d^3\vec{r}\,' \frac{\rho(\vec{r}\,', t)}{|\vec{r} - \vec{r}\,'|}$ (the "instantaneous" Coulomb potential)

and $\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{A} = -\mu_0 \vec{J} + \frac{1}{c^2}\vec{\nabla}\frac{\partial \varphi}{\partial t}$

ecall (from last semester) Helmholtz's theorem that expresses a vector field as a sum of a curlless term and a div-less term, e.g.

$$\vec{J}(\vec{r},t) = \vec{J}_\ell(\vec{r},t) + \vec{J}_t(\vec{r},t)$$

$$\vec{J}_\ell(\vec{r},t) = -\frac{1}{4\pi}\vec{\nabla}\int d^3\vec{r}\,'\,\frac{(\vec{\nabla}\cdot\vec{J})(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}$$

$$\vec{J}_t(\vec{r},t) = \frac{1}{4\pi}\,\vec{\nabla}\times\int d^3\vec{r}\,'\,\frac{(\vec{\nabla}\times\vec{J})(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}$$

now $\frac{\partial\varphi}{\partial t} = \frac{1}{\varepsilon_0}\frac{1}{4\pi}\int d^3\vec{r}\,'\,\frac{1}{|\vec{r}-\vec{r}\,'|}\frac{\partial\rho(\vec{r}\,',t)}{\partial t} = \frac{1}{\varepsilon_0}\frac{1}{4\pi}\int d^3\vec{r}\,'\,\frac{1}{|\vec{r}-\vec{r}\,'|}(-\vec{\nabla}\cdot\vec{J})(\vec{r}\,',t)$ using the continuity eqn $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$

thus $\frac{1}{c^2}\vec{\nabla}\frac{\partial\varphi}{\partial t} = \frac{\varepsilon_0\mu_0}{\varepsilon_0}\left(-\frac{1}{4\pi}\right)\vec{\nabla}\int d^3\vec{r}\,'\,\frac{(\vec{\nabla}\cdot\vec{J})(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|} = \mu_0\vec{J}_\ell$

so $\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{A} = -\mu_0\vec{J}_t$ & it is only the "transverse" (divergence-less) component of the current density that generates $\vec{A}$, in the Coulomb gauge

CONSERVATION LAWS & MAXWELL'S EQUATIONS

consider the work done on a collection of particles with charge density $\rho(\vec r, t)$ and current density $\vec J(\vec r, t) = \rho(\vec r, t)\, \vec v(\vec r, t)$ in a volume V, by electromagnetic fields $\vec E(\vec r, t)$, $\vec B(\vec r, t)$

$$\frac{dW}{dt} = \int_V d^3\vec r \;\; \vec v \cdot \vec f_{em} = \int_V d^3\vec r \; \left(\rho\vec E + \vec J \times \vec B\right)\cdot \vec v = \int_V d^3\vec r \;\; \vec J \cdot \vec E$$

now we can eliminate $\vec J$ in favor of $\vec E, \vec B$ using the Ampère-Maxwell eqn $\vec\nabla\times\vec B = \mu_0 \vec J + \mu_0\varepsilon_0 \frac{\partial}{\partial}$

$$\vec J \cdot \vec E = \frac{1}{\mu_0} \vec\nabla\times\vec B \cdot \vec E - \varepsilon_0\, \vec E \cdot \frac{\partial \vec E}{\partial t}$$

now we have an identity $\vec\nabla\cdot(\vec E\times\vec B) = \underbrace{\vec B\cdot\vec\nabla\times\vec E}_{\text{Faraday} \;\to\; -\frac{\partial \vec B}{\partial t}} - \vec E\cdot\vec\nabla\times\vec B$

$$\Rightarrow \vec J\cdot\vec E = \frac{-1}{\mu_0}\vec B\cdot\frac{\partial \vec B}{\partial t} - \varepsilon_0 \vec E\cdot\frac{\partial\vec E}{\partial t} \;\; \frac{1}{\mu_0}\vec\nabla\cdot(\vec E\times\vec B)$$

& thus

$$\underbrace{\frac{d}{dt}\int_V d^3\vec r \;\; \frac{1}{2}\varepsilon_0\left(\vec E^2 + c^2\vec B^2\right)}_{\substack{\text{total energy in the}\\ \text{electromagnetic field}\\ U_{em}}} = -\underbrace{\int_V d^3\vec r \;\; \vec J\cdot\vec E}_{\substack{\text{rate of}\\ \text{change of}\\ \text{mechanical energy}\\ \text{of particles,}\\ \frac{d}{dt}U_{mech}}} - \int_V d^3\vec r \;\; \frac{1}{\mu_0}\vec\nabla\cdot(\vec E\times\vec B)$$

"Poynting's theorem"

$$\frac{d}{dt}\left(U_{em} + U_{mech}\right) = -\int d^3\vec r \;\; \frac{1}{\mu_0}\vec\nabla\cdot(\vec E\times\vec B)$$

defining the "Poynting vector" $\vec S \equiv \frac{1}{\mu_0}\vec E\times\vec B$

and using the divergence theorem

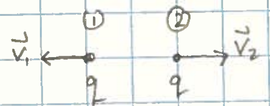
$$\underbrace{\frac{d}{dt}\left(U_{em} + U_{mech}\right)}_{\substack{\text{change in total energy in}\\ \text{volume, V}}} = -\underbrace{\int d\vec A\cdot\vec S}_{\substack{\text{inflow/outflow}\\ \text{of em energy through}\\ \text{the boundary of V.}}} \qquad [E]$$

if we consider the electromagnetic energy density $u_{em} \equiv \frac{1}{2}\varepsilon_0\left(\vec E^2 + c^2\vec B^2\right)$

then $\dfrac{\partial u_{em}}{\partial t} + \vec\nabla\cdot\vec S = -\vec J\cdot\vec E$

So we see that electromagnetic fields can carry energy – this makes us suspect they might also carry momentum. A simple illustration involving two chages shows that they can:

Consider two identical chages in the center of mass frame

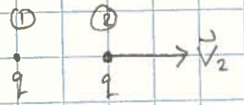


the force on ① due to ② is $\vec F_1 = q\left(\vec E(\vec r_1) + \vec v_1 \times \vec B(\vec r_1)\right)$

the force on ② due to ① is $\vec F_2 = q\left(\vec E(\vec r_2) + \vec v_2 \times \vec B(\vec r_2)\right)$

but the system is symmetric $\vec F_1 = -\vec F_2$, Newton's 3rd law holds

Now consider the same system in the frame of reference where particle ① is instantaneously at rest



In this case there is no magnetic force since $\vec v_1 \times \vec B(\vec r_1) = 0$ $(\vec v_1 = 0)$
$\vec v_2 \times \vec B(\vec r_2) = 0$ since ① doesn't produce a $\vec B$.

$\vec F_1 = q\,\vec E(\vec r_1)$ ↑ field from moving charge

$\vec F_2 = q\,\vec E(\vec r_2)$ ↑ field from static charge

$$\vec F_1 = q\left(-\vec\nabla\varphi_2 - \frac{\partial}{\partial t}\vec A_2\right) \qquad \vec F_2 = q\left(-\vec\nabla\varphi_1\right)$$

and in Coulomb gauge $\varphi_1(\vec r_2) = \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{|\vec r_1 - \vec r_2(t)|}$, $\varphi_2(\vec r_1) = \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{|\vec r_1 - \vec r_2(t)|}$

the relevant gradients of these potentials are equal & opposite

$$\vec\nabla\varphi_1(\vec r_2) = \frac{q}{4\pi\epsilon_0}\frac{\vec r_1 - \vec r_2}{|\vec r_1 - \vec r_2|^3}, \quad \vec\nabla\varphi_2(\vec r_1) = -\frac{q}{4\pi\epsilon_0}\frac{\vec r_1 - \vec r_2}{|\vec r_1 - \vec r_2|^3}$$

So $\vec F_1 + \vec F_2 = -q\dfrac{\partial}{\partial t}\vec A_2(\vec r_1, t)$, but $\dfrac{d\vec P_{12}}{dt} = \vec F_1 + \vec F_2$ so this seems to suggest that the total momentum of the "closed" system of particles ① & ② is not conserved!

The way out is to associate $q\vec A_2(\vec r_1, t)$ with a momentum carried by the electromagnetic field.

et's see if we can derive a more genera result for the linear momentum carried
y an electromagnetic field.

onsider the total mechanical force on the charge & current densities in a volume, V,

$$\vec{F}_{mech} = \int_V d^3\vec{r}\,\left(\rho\vec{E} + \vec{J}\times\vec{B}\right)$$

but since $\rho = \varepsilon_0\vec{\nabla}\cdot\vec{E}$ and $\vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \varepsilon_0\frac{\partial\vec{E}}{\partial t}$

$$\vec{F}_{mech} = \int_V d^3\vec{r}\,\left(\varepsilon_0\,(\vec{\nabla}\cdot\vec{E})\vec{E} + \varepsilon_0\,\vec{B}\times\frac{\partial\vec{E}}{\partial t} + \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B}\right)$$

now
$$\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \vec{E}\times\frac{\partial\vec{B}}{\partial t} + \frac{\partial\vec{E}}{\partial t}\times\vec{B}$$
$$= \vec{E}\times(-\vec{\nabla}\times\vec{E}) - \vec{B}\times\frac{\partial\vec{E}}{\partial t}$$

$$\vec{F}_{mech} = \int_V d^3\vec{r}\,\left(\varepsilon_0(\vec{\nabla}\cdot\vec{E})\vec{E} + \varepsilon_0\left[-\vec{E}\times(\vec{\nabla}\times\vec{E}) - \frac{\partial}{\partial t}(\vec{E}\times\vec{B})\right] + \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B}\right)$$

adding a term $\frac{1}{\mu_0}(\vec{\nabla}\cdot\vec{B})\vec{B}$, which is zero we have the symmetric form

$$= \int_V d^3\vec{r}\,\left(-\varepsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \varepsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} - \vec{E}\times(\vec{\nabla}\times\vec{E})\right] + 1/\mu_0\left[(\vec{\nabla}\cdot\vec{B})\vec{B} - \vec{B}\times(\vec{\nabla}\times\vec{B})\right]\right)$$

now
$$\left[\vec{A}\times(\vec{\nabla}\times\vec{A})\right]_i = \epsilon_{ijk}A_j\,\epsilon_{klm}\,\partial_l A_m = \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)A_j\,\partial_l A_m$$
$$= A_j\,\partial_i A_j - (\vec{A}\cdot\vec{\nabla})A_i$$
$$= \frac{1}{2}\nabla_i(\vec{A}\cdot\vec{A}) - (\vec{A}\cdot\vec{\nabla})A_i$$

$$\vec{F}_{mech} = \int_V d^3\vec{r}\,\left(-\frac{\partial}{\partial t}\left(\vec{S}/c^2\right) + \varepsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E} - \frac{1}{2}\vec{\nabla}(\vec{E}\cdot\vec{E})\right] + 1/\mu_0\left[(\vec{\nabla}\cdot\vec{B})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{B} - \frac{1}{2}\vec{\nabla}(\vec{B}\cdot\vec{B})\right]\right)$$

We can define a rank-two tensor, the MAXWELL STRESS TENSOR,

$$T_{ij} \equiv \varepsilon_0 \left(E_i E_j + c^2 B_i B_j - \frac{1}{2}\delta_{ij}\left(E^2 + c^2 B^2\right)\right)$$

symmetric $T_{ij} = T_{ji}$

(note the trace, $\sum_i T_{ii} = \frac{1}{2}\varepsilon_0\left(E^2 + c^2 B^2\right)$)
$= u_{em}$

then $(\vec{\nabla}\cdot\overleftrightarrow{T})_j \equiv \sum_i \partial_i T_{ij}$ appears in our equation for Fmech

$$= \varepsilon_0 \left((\vec{\nabla}\cdot\vec{E})E_j + (\vec{E}\cdot\vec{\nabla})E_j - \frac{1}{2}\partial_j E^2 \right)$$
$$+ {}^1\!/_{\mu_0}\left(E \to B \right)$$

$$\vec{F}_{mech} = \frac{d}{dt}\vec{P}_{mech} = \int_V d^3\vec{r} \left(-\frac{\partial}{\partial t}\left(\vec{S}/c^2\right) + \vec{\nabla}\cdot\overleftrightarrow{T} \right)$$

we identify $\vec{S}/c^2$ with the ELECTROMAGNETIC MOMENTUM DENSITY

making the total linear momentum carried by the em field in V, $\vec{P}_{em} = \int d^3\vec{r}\ \vec{S}/c^2$

& $\frac{d}{dt}\left(\vec{P}_{mech} + \vec{P}_{em}\right) = \int_V d^3\vec{r}\ \vec{\nabla}\cdot\overleftrightarrow{T}$

and using the divergence theorem $\frac{d}{dt}\left(P^i_{mech} + P^i_{em}\right) = \int dA\ \hat{n}_j T_{ji}$ (j summed)

compare with eqn [E] on page 11.

T_{ij} = rate at which the j^{th} component of momentum flows through an area element $dA\,\hat{n}_i$

DISCRETE SYMMETRIES & ELECTROMAGNETISM

The physics of electromagnetism is INVARIANT under a number of symmetry transformations. Some of these, like rotations, are continuous – we will delay considering these until we're ready to consider Lorentz transformations and special relativity. Others are DISCRETE transformations: SPATIAL INVERSION ("parity") $\vec{r} \xrightarrow{P} -\vec{r}$
and TIME REVERSAL $t \xrightarrow{T} -t$

PARITY:

vector quantities can have one of two possible behaviours under the parity operation

(POLAR) VECTORS $\vec{V} \xrightarrow{P} -\vec{V}$ (e.g. $\vec{r}$; $\vec{p}$)

AXIAL VECTORS $\vec{A} \xrightarrow{P} \vec{A}$ (e.g. $\vec{L} = \vec{r} \times \vec{p}$)

Similarly scalar quantities:

(TRUE) SCALAR $S \xrightarrow{P} S$ (e.g. r^2, p^2, $\vec{r} \cdot \vec{p}$)

PSEUDOSCALAR $\mathcal{P} \xrightarrow{P} -\mathcal{P}$ (e.g. $\vec{V}_1 \cdot \vec{V}_2 \times \vec{V}_3$ for 3 (polar) vectors)

TIME REVERSAL:

all quantities are either even or odd under time reversal, $X \xrightarrow{T} \pm T$ $\begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}$

electromagnetic quantities:

we propose that electric charge is a (true) scalar, then charge density ρ is also a scalar. Since $\vec{\nabla}$ is a (polar) vector, Gauss's law

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$$

implies that $\vec{E}$ must be a (polar) vector if electromagnetism is invariant under parity transforms, which we experimentally find it to be.

Faraday's law, $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, then tell's us that $\vec{B}$ is an axial vector, which

we can also see using the static Ampère's law $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ noting that $\vec{J}$, which we can express as $\rho \vec{v}$ is a (polar) vector.

These equations also show us that $\vec{E}$ is even under time-reversal
while $\vec{B}$ is odd under time-reversal

$$\begin{tikzcd}
 & \rho(-\vec{r},t) \\
\rho(\vec{r},t) \arrow[ur, "P"] \arrow[dr, "T"'] & \\
 & \rho(\vec{r},-t)
\end{tikzcd}$$

$$\begin{tikzcd}
 & -\vec{J}(-\vec{r},t) \\
\vec{J}(\vec{r},t) \arrow[ur, "P"] \arrow[dr, "T"'] & \\
 & -\vec{J}(\vec{r},-t)
\end{tikzcd}$$

$$\begin{tikzcd}
 & -\vec{E}(-\vec{r},t) \\
\vec{E}(\vec{r},t) \arrow[ur, "P"] \arrow[dr, "T"'] & \\
 & \vec{E}(\vec{r},-t)
\end{tikzcd}$$

$$\begin{tikzcd}
 & \vec{B}(-\vec{r},t) \\
\vec{B}(\vec{r},t) \arrow[ur, "P"] \arrow[dr, "T"'] & \\
 & -\vec{B}(\vec{r},t)
\end{tikzcd}$$

$$\begin{tikzcd}
 & -\vec{S}(-\vec{r},t) \\
,t) = \vec{E}\times\vec{B} \arrow[ur, "P"] \arrow[dr, "T"'] & \\
 & -\vec{S}(\vec{r},-t)
\end{tikzcd}$$

$$\begin{tikzcd}
 & u_{em}(-\vec{r},t) \\
u_{em}(\vec{r},t) = E^2+c^2B^2 \arrow[ur, "P"] \arrow[dr, "T"'] & \\
 & u_{em}(\vec{r},-t)
\end{tikzcd}$$