FIELDS FROM MOVING CHARGES (bremsstrahlung & synchrotron radiation)

The approach of Liénard & Wiechert leads us to expressions for the fields from a single moving charge via potentials in the Lorenz gauge.

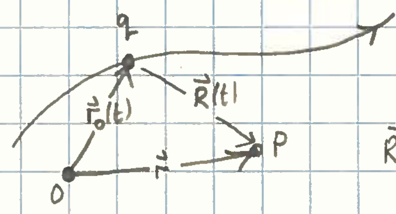
We previously showed that in Lorenz gauge

$$\varphi(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int d^3\vec{r}\,'\,\frac{1}{|\vec{r}-\vec{r}\,'|}\,\rho(\vec{r}\,', t-\tfrac{1}{c}|\vec{r}-\vec{r}\,'|)$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3\vec{r}\,'\,\frac{1}{|\vec{r}-\vec{r}\,'|}\,\vec{J}(\vec{r}\,', t-\tfrac{1}{c}|\vec{r}-\vec{r}\,'|)$$

with the sources evaluated at the retarded time, $t' = t-\frac{1}{c}|\vec{r}-\vec{r}\,'|$

For a point charge following a trajectory $\vec{r}_0(t)$



$\rho(\vec{r},t) = q\,\delta(\vec{r}-\vec{r}_0(t))$

$\vec{J}(\vec{r},t) = q\,\vec{v}(t)\,\delta(\vec{r}-\vec{r}_0(t))$

$\vec{R}(t) = \vec{r}-\vec{r}_0(t) = R(t)\,\hat{n}(t)$

Let's consider $\varphi(\vec{r},t)$ (since $\vec{A}(\vec{r},t)$ can be obtained by analogous methods), and write the expression so that the retarded Green's function is visible:

$$\varphi(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int d^3\vec{r}\,'\int dt'\,\frac{\rho(\vec{r}\,',t')}{|\vec{r}-\vec{r}\,'|}\,\delta\left[t'-t+\tfrac{1}{c}|\vec{r}-\vec{r}\,'|\right]$$

and inserting the charge density of the moving point charge & performing the space integral

$$\varphi(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\int dt'\,\frac{1}{R(t')}\,\delta[t'-t+\tfrac{1}{c}R(t')] \qquad [F]$$

and the remaining integral can be evaluated using the identity $\delta[f(x)] = \sum_n \frac{1}{\left|\frac{df}{dx}\right|_{x_n}}\delta(x-x_n)$

where $\frac{df}{dx}$ is being evaluated at the zeroes of $f(x)$.

$f(t') = t' - (t-\frac{1}{c}R(t')) = 0 \quad / \quad \frac{df}{dt'} = 1 + \frac{1}{c}\frac{d}{dt'}R(t')$

$R(t')^2 = (\vec{r}-\vec{r}_0(t'))\cdot(\vec{r}-\vec{r}_0(t')) = r^2 - 2\vec{r}\cdot\vec{r}_0(t') + r_0^2(t')$

$$2R(t')\frac{d}{dt'}R(t') = -2\vec{r}\cdot\vec{v}(t') \quad +2\vec{r}_0\cdot\vec{v} \qquad = -2\vec{v}\cdot(\vec{r}-\vec{r}_0(t')) = -2\,\vec{v}\cdot\vec{R}$$

$$\frac{d}{dt'}R(t') = -\,\vec{v}\cdot\hat{n}(t')$$

$$\& \quad \frac{df}{dt'} = 1-\vec{\beta}(t')\cdot\hat{n}(t')$$

the solution of $t'-t-\frac{1}{c}R(t')=0$ is the retarded time, t_{ret}.

and hence

$$\varphi(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\;\frac{1}{R(t_{ret})}\;\frac{1}{1-\vec{\beta}\cdot\hat{n}(t_{ret})} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{R-\vec{\beta}\cdot\vec{R}}\right]_{ret}.$$

& for the vector potential $\vec{A}(\vec{r},t) = \frac{\mu_0\,q}{4\pi\varepsilon_0}\;\frac{\vec{v}(t_{ret})}{R(t_{ret})}\;\frac{1}{1-\vec{\beta}\cdot\hat{n}(t_{ret})} = \frac{\mu_0\,q}{4\pi}\left[\frac{\vec{v}}{R-\vec{\beta}\cdot\vec{R}}\right]_{ret}.$

the electric and magnetic fields are most easily obtained by returning to expressions like [F] for the scalar potential

$$\vec{E}(\vec{r},t) = -\vec{\nabla}\varphi(\vec{r},t) - \frac{\partial \vec{A}(\vec{r},t)}{\partial t}$$

$$= -\frac{q}{4\pi\epsilon_0}\vec{\nabla}_{\vec{r}}\int dt' \frac{1}{R(t')}\delta[t'-t+\tfrac{1}{c}R(t')] - \frac{\mu_0 q}{4\pi}\frac{\partial}{\partial t}\int dt' \frac{\vec{v}(t')}{R(t')}\delta[t'-t+\tfrac{1}{c}R(t')]$$

$$\vec{R} = \vec{r}-\vec{r}_0 = \hat{n}R \qquad \text{so} \quad R = \hat{n}\cdot(\vec{r}-\vec{r}_0) \qquad \& \quad \vec{\nabla}_{\vec{r}}R = \hat{n}$$

$$\vec{\nabla}_{\vec{r}}\frac{1}{R} = -\frac{1}{R^2}\vec{\nabla}_{\vec{r}}R = -\frac{\hat{n}}{R^2} = -\vec{R}/R^3$$

$$\text{so} \quad \vec{\nabla}_{\vec{r}}\delta\left[t'-t+\tfrac{1}{c}R(t')\right] = \frac{d}{df}\delta[f]\,\vec{\nabla}_{\vec{r}}f \qquad f = t'-t+\tfrac{1}{c}R$$

$$= -\hat{n}\frac{1}{c}\frac{\partial}{\partial t}\delta\left[t'-t+\tfrac{1}{c}R(t')\right]$$

$$\Rightarrow \vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\int dt'\left(\frac{\hat{n}}{R^2} + \hat{n}\frac{1}{cR}\frac{\partial}{\partial t} - \frac{1}{c^2}\frac{\vec{v}}{R}\frac{\partial}{\partial t}\right)\delta\left[t'-t+\tfrac{1}{c}R(t')\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\int dt'\frac{\hat{n}}{R^2}\delta[t'-t+R(t')/c] + \frac{1}{c}\frac{\partial}{\partial t}\int dt'\frac{\hat{n}-\vec{\beta}}{R}\delta[t'-t+\tfrac{1}{c}R(t')]\right]$$

& the same technique we used to evaluate [F] can be used to obtain

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\left[\frac{\hat{n}}{R^2(1-\hat{n}\cdot\vec{\beta})}\right]_{ret} + \frac{q}{4\pi\epsilon_0}\frac{1}{c}\frac{d}{dt}\left[\frac{\hat{n}-\vec{\beta}}{R(1-\hat{n}\cdot\vec{\beta})}\right]_{ret}$$

sly

$$\vec{B}(\vec{r},t) = \frac{\mu_0 q}{4\pi}\left[\frac{\vec{v}\times\hat{n}}{R^2(1-\hat{n}\cdot\vec{\beta})}\right]_{ret} + \frac{\mu_0 q}{4\pi}\frac{1}{c}\frac{d}{dt}\left[\frac{\vec{v}\times\hat{n}}{R(1-\hat{n}\cdot\vec{\beta})}\right]_{ret}$$

but we can manipulate these into a more useful form...

$$t_{ret} - t + \frac{1}{c} R(t_{ret}) = 0 \qquad \& \qquad \frac{dR}{dt} = -\vec{v}\cdot\hat{n} = -c\,\vec{\beta}\cdot\hat{n}$$

$$\downarrow$$

$$1 - \frac{dt}{dt_{ret}} + \frac{1}{c}\frac{dR_{ret}}{dt_{ret}} = 0$$

$$1 - \frac{dt}{dt_{ret}} - \vec{\beta}\cdot\hat{n}\Big|_{ret} = 0 \qquad \Rightarrow \qquad \underline{\frac{dt}{dt_{ret}} = \left(1 - \vec{\beta}\cdot\hat{n}\right)_{ret}}$$

$$\frac{d}{dt}\hat{n} = \frac{d}{dt}\frac{\vec{R}}{R} = \frac{1}{R}\frac{d\vec{R}}{dt} - \frac{\vec{R}}{R^2}\frac{dR}{dt} = -\vec{v}/_R - \frac{\vec{R}}{R^2}\left(-\vec{v}\cdot\hat{n}\right)$$

$$= -c\frac{\vec{\beta}}{R} + \frac{c}{R}\hat{n}(\hat{n}\cdot\beta) = \frac{c}{R}\left(\hat{n}(\hat{n}\cdot\vec{\beta}) - (\hat{n}\cdot\hat{n})\vec{\beta}\right) = \frac{c}{R}\hat{n}\times(\hat{n}\times\vec{\beta})$$

$$\underline{\frac{d}{dt}\left(1 - \vec{\beta}\cdot\hat{n}\right) = -\vec{\beta}\cdot\frac{d\hat{n}}{dt} - \hat{n}\cdot\frac{d\vec{\beta}}{dt}}$$

After some algebra we obtain

$$\vec{E}(\vec{r},t) = \underbrace{\frac{q}{4\pi\epsilon_0}\left[\frac{(\hat{n}-\vec{\beta})(1-\beta^2)}{(1-\hat{n}\cdot\vec{\beta})^3\,R^2}\right]_{ret}}_{\text{"}\vec{E}_{vel}\text{"}} + \underbrace{\frac{q}{4\pi\epsilon_0}\frac{1}{c}\left[\frac{\hat{n}\times\left((\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\right)}{(1-\hat{n}\cdot\vec{\beta})^3\,R}\right]_{ret}}_{\text{"}E_{acc}\text{"}}$$

N.B. this is a radiation field $\sim \frac{1}{R}$.

$$\& \quad \vec{B}(\vec{r},t) = \underbrace{\frac{\mu_0 q}{4\pi}\left[\frac{(\vec{v}\times\hat{n})(1-\beta^2)}{(1-\hat{n}\cdot\vec{\beta})^3\,R^2}\right]_{ret}}_{\text{"}\vec{B}_{vel}\text{"}} + \underbrace{\frac{\mu_0 q}{4\pi}\left[\frac{(\vec{\beta}\times\hat{n})(\dot{\vec{\beta}}\cdot\hat{n}) + (1-\hat{n}\cdot\vec{\beta})\dot{\vec{\beta}}\times\hat{n}}{(1-\hat{n}\cdot\vec{\beta})^3\,R}\right]_{ret}}_{\text{"}\vec{B}_{acc}\text{"}}$$

$$\hat{n}_{ret}\times\vec{E}_{vel} = c\vec{B}_{vel}$$

$$\hat{n}_{ret}\times\vec{E}_{acc} = c\vec{B}_{acc}$$

the radiation field has a Poynting vector

$$\vec{S}(t) = \frac{1}{\mu_0}\vec{E}_{acc} \times \vec{B}_{acc} = \frac{1}{\mu_0 c}\vec{E}_{acc} \times (\hat{n}_{ret} \times \vec{E}_{acc}) = \frac{\hat{n}_{ret}}{\mu_0 c}|\vec{E}_{acc}|^2$$

and the angular distribution of emitted power is

$$\frac{dP(t_{ret})}{d\Omega} = \frac{dU}{dt_{ret}\,d\Omega} = \frac{dt}{dt_{ret}}\frac{dU}{dt\,d\Omega} = (1-\hat{n}\cdot\vec{\beta})_{ret} \cdot R^2\vec{S}(t)\cdot\hat{n}_{ret}$$

$$= (1-\hat{n}\cdot\vec{\beta})_{ret} \cdot \left(\frac{q}{4\pi\varepsilon_0 c}\right)^2 \cdot \frac{1}{\mu_0 c} \cdot \left[\frac{|\hat{n}\times((\hat{n}-\vec{\beta})\times\dot{\vec{\beta}})|^2}{(1-\hat{n}\cdot\vec{\beta})^6}\right]_{ret}$$

$$= \frac{q^2}{16\pi^2\varepsilon_0 c}\left.\frac{|\hat{n}\times((\hat{n}-\vec{\beta})\times\dot{\vec{\beta}})|^2}{(1-\hat{n}\cdot\vec{\beta})^5}\right|_{ret}$$

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* non-relativistic motion, $\vec{\beta}\to 0$, $\dot{\vec{\beta}} = c\vec{a} = ca\hat{z}$ & say that at this moment the particle is at the origin so $\hat{n}=\hat{r}$

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0 c}|\hat{r}\times(\hat{r}\times ca\hat{z})|^2$$

$$= \frac{q^2}{16\pi^2\varepsilon_0 c}c^2a^2|\hat{r}(\hat{r}\cdot\hat{z}) - \hat{z}|^2 = \frac{q^2ca^2}{16\pi^2\varepsilon_0}\left((\hat{r}\cdot\hat{z})^2 - 2(\hat{r}\cdot\hat{z})^2 + 1\right)$$

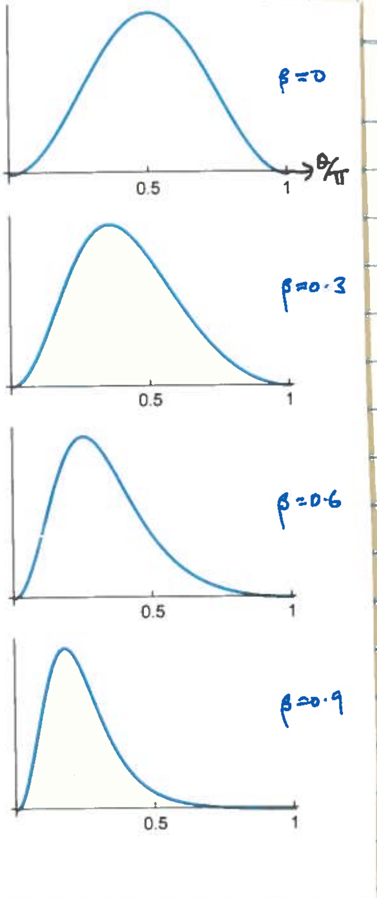
$$= \frac{q^2ca^2}{16\pi^2\varepsilon_0}(1-\cos^2\theta) = \underline{\frac{q^2ca^2}{16\pi^2\varepsilon_0}\sin^2\theta}$$

$\Rightarrow$ most power in perpendicular direction

* acceleration in the direction of motion

$$\dot{\vec{\beta}} = c a \hat{z} \qquad \frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0 c}\frac{1}{(1-\beta\cos\theta)^5} c^2 a^2 \sin^2\theta$$

$$\vec{\beta} = c v \hat{z} \qquad\qquad = \frac{q^2 c a^2}{16\pi^2\varepsilon_0}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}$$



"bremsstrahlung" → "breaking radiation"

increasingly peaked in forward direction as $\beta \to 1$

for $\beta \simeq 1$, $1-\beta = \dfrac{1-\beta^2}{1+\beta} = \dfrac{1}{\gamma^2}\dfrac{1}{1+\beta} \longrightarrow \dfrac{1}{2\gamma^2}$

& in the forward direction, $\cos\theta \simeq 1 - \frac{1}{2}\theta^2$

so $1-\beta\cos\theta \longrightarrow 1-\beta + \frac{1}{2}\beta\theta^2 \simeq \dfrac{1}{2\gamma^2} + \dfrac{1}{2}\theta^2$

$$\longrightarrow \frac{1+\gamma^2\theta^2}{2\gamma^2}$$

$$\frac{dP}{d\Omega} \longrightarrow \frac{q^2 c a^2}{16\pi^2\varepsilon_0}\left(\frac{2\gamma^2}{1+\gamma^2\theta^2}\right)^5 \theta^2 = \frac{2q^2 c a^2}{\pi^2\varepsilon_0}\gamma^8 \frac{(\gamma\theta)^2}{(1+(\gamma\theta)^2)^5}$$

$f(x) = \dfrac{x^2}{(1+x^2)^5}$ $\qquad 0 = \dfrac{df}{dx} = \dfrac{2x}{(1+x^2)^5} - 5\dfrac{x^2}{(1+x^2)^6}(+2x) = (1+x^2) - 5x^2 = 1-4x^2$

$\Rightarrow x_{max} = 1/2 \qquad \Rightarrow \theta_{max} = 1/2\gamma$ & width of peak $\sim \frac{1}{\gamma}$

$$\left.\frac{dP}{d\Omega}\right|_{\theta_{max}} \propto \gamma^8$$

e^-, $\gamma \sim 20$
$\gamma^8 \sim 2.6\times10^{10}$!

* accn perpendicular to velocity (ie momentary circular motion)

$\vec{v} = v\,\hat{z}$

$\vec{a} = a\,\hat{x}$

$$\hat{n}_{ret} \simeq \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\rightarrow \frac{dP}{d\Omega} = \frac{q^2 c a^2}{16\pi^2 \varepsilon_0} \frac{1}{(1-\beta\cos\theta)^3}\left[1 - \frac{1-\beta^2}{(1-\beta\cos\theta)^2}\sin^2\theta\cos^2\phi\right]$$

& in the limit $\beta \to 1$, $1-\beta^2 = \frac{1}{\gamma^2}$

$$\frac{1}{1-\beta\cos\theta} \rightarrow \frac{2\gamma^2}{1+\gamma^2\theta^2}$$

$$\frac{dP}{d\Omega} \rightarrow \frac{q^2 c a^2}{16\pi^2\varepsilon_0}\frac{8\gamma^6}{(1+\gamma^2\theta^2)^3}\left[1 - \frac{4\gamma^2\theta^2}{(1+\gamma^2\theta^2)^2}\cos^2\phi\right]$$

again strongly peaked in the forward directi

"synchrotron radiation"