PLANE WAVES IN VACUUM & IN SIMPLE MEDIA

Consider the Maxwell equations in a simple medium with no free charges or currents:

$$\vec{\nabla}\cdot\vec{H}=0 \qquad \vec{\nabla}\times\vec{E} = -\mu\frac{\partial \vec{H}}{\partial t}$$
$$\vec{\nabla}\cdot\vec{E}=0 \qquad \vec{\nabla}\times\vec{H} = \varepsilon\frac{\partial \vec{E}}{\partial t} \qquad [M]$$

Taking the curl of the right most equations

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} = -\mu\frac{\partial}{\partial t}\vec{\nabla}\times\vec{H} = -\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} \qquad \left[\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right]\vec{E}=0$$

$$\text{\& similarly} \qquad \left[\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right]\vec{H}=0$$

where $v^2 \equiv 1/\varepsilon\mu$ & in vacuum, $\varepsilon\to\varepsilon_0$, $\mu\to\mu_0$, $v^2\to c^2 = 1/\varepsilon_0\mu_0$

the "refractive index" can be defined, $n = \dfrac{c}{v} = \sqrt{\dfrac{\varepsilon\mu}{\varepsilon_0\mu_0}}$

The fact that we've obtained the free wave eqns for $\vec{E}$ & $\vec{H}$ indicates that we already know one class of solutions, those which depend only upon time and distance in a particular direction,

$$\vec{E}(z,t) \ \& \ \vec{B}(z,t),$$

then we know that each cartesian component of, say, $\vec{E}$, has general solution $f(z+vt)+g(z-vt)$ where f,g are arbitrary functions.

But the Maxwell equations impose further constraints upon these solutions:

$$0 = \vec{\nabla}\cdot\vec{E}(z,t) = \frac{\partial E_z}{\partial z} \Rightarrow E_z(z,t) = fn(t)$$
$$0 = \varepsilon\frac{\partial E_z}{\partial t} - \hat{z}\cdot\vec{\nabla}\times\vec{H} \Rightarrow \frac{\partial E_z}{\partial t} = fn(z)$$

$\Big\}$ E_z = an uninteresting constant which we'll set to zero

It follows that $\vec{E}$ is always perpendicular to $\hat{z}$.

We write the two linearly independent possibilities as

$$\vec{E}_+(z,t) = \vec{f}_\perp(z+vt)$$
$$\vec{E}_-(z,t) = \vec{g}_\perp(z-vt)$$

he magnetic fields are related to these by the Maxwell equations

$$\frac{\partial \vec{H}}{\partial t} = -\frac{1}{\mu}\vec{\nabla}\times\vec{E}$$

& recalling that $E_z = 0$ & $E_\pm$ are not functions of x, y

$$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \partial/\partial z \\ E_x & E_y & 0 \end{vmatrix} = -\hat{x}\frac{\partial}{\partial z}E_y + \hat{y}\frac{\partial}{\partial z}E_x$$

$$= +\hat{z}\times\frac{\partial}{\partial z}\vec{E}$$

so $$\frac{\partial}{\partial t}\begin{Bmatrix}\vec{H}_+ \\ \vec{H}_-\end{Bmatrix} = -\frac{1}{\mu}\hat{z}\times\frac{\partial}{\partial z}\begin{Bmatrix}\vec{f}_\perp(z+vt) \\ \vec{g}_\perp(z-vt)\end{Bmatrix}$$

but $$\left(\frac{\partial}{\partial z} - \frac{1}{v}\frac{\partial}{\partial t}\right)f(z+vt) = 0 \quad \& \quad \left(\frac{\partial}{\partial z} + \frac{1}{v}\frac{\partial}{\partial t}\right)f(z-vt) = 0$$

so $$\frac{\partial}{\partial t}\begin{Bmatrix}\vec{H}_+ \\ \vec{H}_-\end{Bmatrix} = -\frac{1}{\mu v}\hat{z}\times\frac{\partial}{\partial t}\begin{Bmatrix}\vec{f}_\perp(z+vt) \\ -\vec{g}_\perp(z-vt)\end{Bmatrix}$$

and integrating w.r.t time and dropping the uninteresting time-independent term we have

$$\vec{H}_+ = -\frac{1}{\mu v}\hat{z}\times\vec{f}_\perp(z+vt) \qquad \vec{E}_+ = \vec{f}_\perp(z+vt)$$

$$\vec{H}_- = +\frac{1}{\mu v}\hat{z}\times\vec{g}_\perp(z-vt) \qquad \vec{E}_- = \vec{g}_\perp(z-vt)$$

Clearly $\vec{E}_+\cdot\vec{H}_+ = 0$ & $\vec{E}_-\cdot\vec{H}_- = 0$ so for each possibility we have three mutually
$(H_+)_z = 0 \qquad (H_-)_z = 0$ perpendicular vectors $(\hat{z}, \vec{E}, \vec{H})$

"transverse electromagnetic plane wave"

There's nothing special in the choice of the $\hat{z}$ axis for propagation - choosing a general direction, $\hat{k}$, we may write

$$\vec{E}(\vec{r},t) = \vec{E}_\perp(\vec{k}\cdot\vec{r} - vkt) \quad \& \quad \mu v\vec{H} = \hat{k}\times\vec{E}(\vec{r},t)$$

note that in the vacuum case $c|\vec{B}| = |\vec{E}|$

the flow of energy and momentum for a plane wave solution can be considered - for simplicity we'll consider a wave in vacuum

the energy density $u_{em} = \frac{1}{2}\varepsilon_0 (\vec{E}^2 + c^2\vec{B}^2) = \frac{1}{2}\varepsilon_0 \left(\vec{E}_\perp \cdot \vec{E}_\perp + (\hat{k}\times\vec{E}_\perp)\cdot(\hat{k}\times\vec{E}_\perp) \right)$

the second term can be rewritten $\hat{k}\cdot(\vec{E}_\perp \times (\hat{k}\times\vec{E}_\perp))$

$$= \hat{k}\cdot(\hat{k}\,\vec{E}_\perp\cdot\vec{E}_\perp - \vec{E}_\perp\,\hat{k}\cdot\vec{E}_\perp)$$

$$= \vec{E}_\perp\cdot\vec{E}_\perp \quad \text{since } \hat{k}\cdot\vec{E}_\perp = 0.$$

thus $u_{em} = \varepsilon_0 |\vec{E}_\perp|^2$

the Poynting vector $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\frac{1}{c}\vec{E}_\perp \times (\hat{k}\times\vec{E}_\perp) = \hat{k}\,\frac{1}{c\mu_0}|\vec{E}_\perp|^2 = u_{em}\, c\,\hat{k}$

so energy (& momentum, $\vec{S}/c^2$) is carried at speed c in the direction of propagation ($\hat{k}$) of the wave.

monochromatic plane waves – (in vacuum)

any function $f(\vec{k}\cdot\vec{r} - ckt)$ can be expanded in Fourier modes – we can consider each one of these as a "monochromatic plane wave.

We write
$$\vec{E}(\vec{r},t) = Re\left\{ \vec{\mathcal{E}}_\perp e^{i(\vec{k}\cdot\vec{r}-\omega t)} \right\}$$
$$\vec{B}(\vec{r},t) = Re\left\{ \vec{\mathcal{B}}_\perp e^{i(\vec{k}\cdot\vec{r}-\omega t)} \right\}$$
with $\vec{\mathcal{E}}_\perp, \vec{\mathcal{B}}_\perp$ complex & $\omega = c|\vec{k}|$ in vacuum

but we'll tend to not write the Real part explicitly usually. One place we need to be careful about dropping the Re is when expressions are not linear in $\vec{E}$ or $\vec{B}$, e.g.

$$u_{em} = \frac{1}{2}\varepsilon_0\left[\left(Re(\vec{E})\right)^2 + c^2\left(Re\,\vec{B}\right)^2 \right]$$

$Re\,\vec{E} = \frac{1}{2}(\vec{E} + \vec{E}^*)$ so
$$u_{em} = \frac{1}{8}\varepsilon_0\left[\left(\vec{\mathcal{E}}_\perp e^{i\phi} + \vec{\mathcal{E}}_\perp^* e^{-i\phi}\right)^2 + \dots \right]$$
$$= \frac{1}{8}\varepsilon_0\left[2\vec{\mathcal{E}}_\perp\cdot\vec{\mathcal{E}}_\perp^* + \vec{\mathcal{E}}_\perp\cdot\vec{\mathcal{E}}_\perp e^{2i(\vec{k}\cdot\vec{r}+\omega t)} + \vec{\mathcal{E}}_\perp^*\cdot\vec{\mathcal{E}}_\perp^* e^{-2i(\vec{k}\cdot\vec{r}-\omega t)} + \dots \right]$$

but $\frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt\, e^{2i\omega t} = \frac{\omega}{2\pi}\frac{1}{2i\omega}\left[e^{4\pi i} - 1\right] = 0$ so the time-dependant terms average to zero over one cycle of oscillation

$$\& \quad \langle u_{em}\rangle = \frac{1}{4}\varepsilon_0\left(|\vec{\mathcal{E}}_\perp|^2 + c^2|\vec{\mathcal{B}}_\perp|^2\right)$$
but $c|\vec{\mathcal{B}}_\perp| = |\vec{\mathcal{E}}_\perp|$

$$\langle u_{em}\rangle = \frac{1}{2}\varepsilon_0|\vec{\mathcal{E}}_\perp|^2$$

& similarly $\langle\vec{S}\rangle = \langle u_{em}\rangle c\,\hat{k}$

polarization

Let's prove that in general, the tip of the $\vec{E}$ vector traces out an ellipse over the course of one complete cycle of a monochromatic wave.

First define two perpendicular unit vectors $\hat{e}_1$ and $\hat{e}_2$ that are also perpendicular to $\hat{k}$

$$\hat{e}_1 \cdot \hat{e}_2 = 0 \quad \& \quad \hat{e}_1 \times \hat{e}_2 = \hat{k}$$

then $\vec{E}(\vec{r},t) = \mathrm{Re}\left\{\left(\mathcal{E}_1 \hat{e}_1 + \mathcal{E}_2 \hat{e}_2\right) e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\}$

and let's describe $\mathcal{E}_1$ & $\mathcal{E}_2$ in terms of a magnitude and a phase $\mathcal{E}_1 = A_1 e^{i\delta_1}$

$\mathcal{E}_2 = A_2 e^{i\delta_2}$

and we have $E_1 \hat{e}_1 + E_2 \hat{e}_2 = \hat{e}_1 A_1 \cos(\vec{k}\cdot\vec{r} - \omega t + \delta_1) + \hat{e}_2 A_2 \cos(\vec{k}\cdot\vec{r} - \omega t + \delta_2)$

then $\frac{E_1}{A_1} = \cos(\vec{k}\cdot\vec{r} - \omega t + \delta_1)$ & $\frac{E_2}{A_2} = \cos(\vec{k}\cdot\vec{r} - \omega t + \delta_2)$

a little manipulation shows that $\left(\frac{E_1}{A_1}\right)^2 + \left(\frac{E_2}{A_2}\right)^2 - 2\left(\frac{E_1}{A_1}\right)\left(\frac{E_2}{A_2}\right)\cos\delta = \sin^2\delta$ [E] $\delta \equiv \delta_2 - \delta_1$

which is the equation describing an ellipse

→ a special case has $\delta = \pi$ (E_1, E_2 180° out of phase)

then $\vec{E}(\vec{r},t) = \left(A_1 \hat{e}_1 + A_2 \hat{e}_2\right) \cos(\vec{k}\cdot\vec{r} - \omega t + \delta_1)$ and $\vec{E}$ points in the $\frac{A_1\hat{e}_1 + A_2\hat{e}_2}{A_1^2 + A_2^2}$ direction oscillating in magnitude

this is called LINEAR POLARIZATION for obvious reasons.

→ special case of $A_1 = A_2 = A/\sqrt{2}$ & $\delta = \pm\pi/2$ (90° out of phase)

then [E] becomes $E_1^2 + E_2^2 = A^2$ which describes a circle of radius A.

and we speak of CIRCULAR POLARIZATION.

e.g. $\delta_1 = 0$, at $\vec{r} = 0$, $\vec{E}_\pm = \frac{A}{\sqrt{2}}\left[\hat{e}_1 \cos\omega t \pm \hat{e}_2 \sin\omega t\right]$

A useful change of basis allows arbitrary elliptical polarization to be expressed as a superposition of circular polarizations:

define $\hat{e}_\pm = \frac{1}{\sqrt{2}}(\hat{e}_1 \pm i\hat{e}_2)$ $\quad$ [N.B. $\hat{e}_- = \hat{e}_+^*$]

then $\hat{e}_\pm^* \cdot \hat{e}_\pm = 1$ $\quad$ $\hat{e}_\pm^* \cdot \hat{e}_\mp = 0$

So then circularly polarized waves may be written

$$\vec{E}_\pm(\vec{r},t) = A\,\hat{e}_\pm\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$ (remembering that we must take the real part)

In this basis a general plane wave may be written

$$\vec{E}(\vec{r},t) = \left(\mathcal{E}_+ \hat{e}_+ + \mathcal{E}_- \hat{e}_-\right) e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

monochromatic plane waves in simple media

$$\vec{E}(\vec{r},t) = \vec{E}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \qquad \vec{H}(\vec{r},t) = \vec{H}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \quad \xrightarrow[\text{eqns}]{\text{maxwell}} \quad \boxed{\vec{k}\cdot\vec{E}=0 \quad \vec{k}\cdot\vec{H}=0} \qquad \vec{k}\times\vec{E} = \omega\mu\vec{H} \qquad \vec{k}\times\vec{H} = -\omega\varepsilon\vec{E}$$

→ "transverse em waves"

$$\frac{|\vec{E}|}{|\vec{H}|} = \frac{|\vec{k}\times\vec{E}|}{|\vec{k}\times\vec{H}|} = \frac{\mu}{\varepsilon}\frac{|\vec{H}|}{|\vec{E}|} \Rightarrow \frac{|\vec{E}|}{|\vec{H}|} = \sqrt{\frac{\mu}{\varepsilon}} \equiv Z$$

"wave impedance"

$$\vec{k}\times(\vec{k}\times\vec{E}) = \omega\mu\,\vec{k}\times\vec{H} = -\omega^2\mu\varepsilon\vec{E}$$

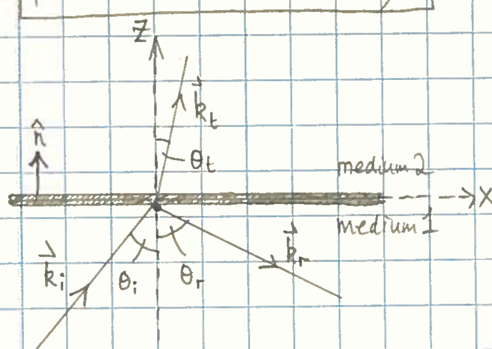
$$\vec{k}(\vec{k}\cdot\vec{E}) - |\vec{k}|^2\vec{E}$$

$$|\vec{k}|^2 = \mu\varepsilon\omega^2 \Rightarrow \omega = vk = \frac{c}{n}k$$

"dispersion relation"

$$\boxed{\vec{k} = n\frac{\omega}{c}\hat{k}, \quad \vec{k}\cdot\vec{E} = 0, \quad Z\vec{H} = \hat{k}\times\vec{E}}$$

reflection and refraction of plane-waves at a boundary



$$\vec{E}_1(\vec{r},t) = \vec{E}_i\, e^{i(\vec{k}_i\cdot\vec{r}-\omega_i t)} + \vec{E}_r\, e^{i(\vec{k}_r\cdot\vec{r}-\omega_r t)}$$

$$\vec{E}_2(\vec{r},t) = \vec{E}_t\, e^{i(\vec{k}_t\cdot\vec{r}-\omega_t t)}$$

& corresponding $\vec{H}$-fields

matching conditions include

$$\hat{n}\times(\vec{E}_1-\vec{E}_2)=0 \quad \& \quad \hat{n}\times(\vec{H}_1-\vec{H}_2)=0$$

i.e. continuity of the components parallel to the interface

the matching conditions can only be satisfied for all times & all points on the boundary ($z=0$) if the phases of the incident, reflected and transmitted waves are the same

$$\Rightarrow \omega_i = \omega_r = \omega_t = \omega \quad \& \quad \vec{k}_i\cdot(x,y,0) = \vec{k}_r\cdot(x,y,0) = \vec{k}_t\cdot(x,y,0)$$

$$\Rightarrow k_{ix} = k_{rx} = k_{tx} \quad \& \quad k_{iy} = k_{ry} = k_{ty}$$

← we can put all three $\vec{k}$ in the xz plane

$k_{ix} = k_{rx}$

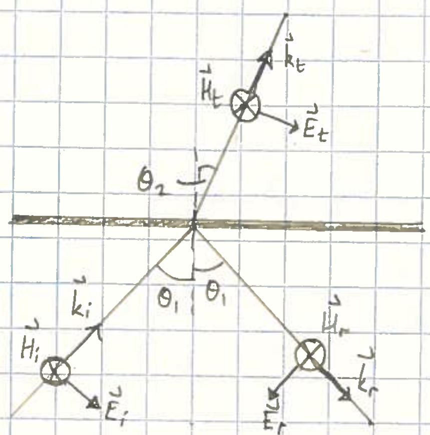
$n_1\frac{\omega}{c}\sin\theta_i = \frac{n_1\omega}{c}\sin\theta_r$

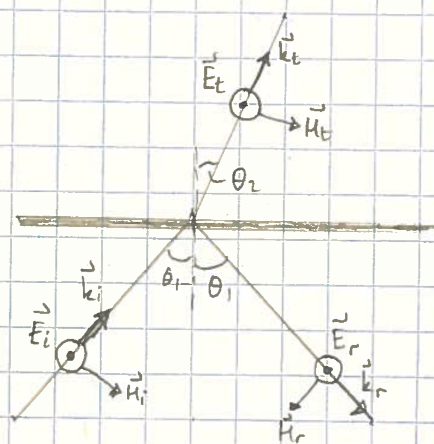
$k_{ix} = k_{tx}$

$n_1\frac{\omega}{c}\sin\theta_i = \frac{n_2\omega}{c}\sin\theta_t$

$\boxed{n_1\sin\theta_1 = n_2\sin\theta_2}$ Snell's law

We can find the behavior of the field amplitudes by applying the matching conditions —
this can best be done by separating in to a bases where one component has $\vec{E}$ parallel to the interface and the other has $\vec{E}$ perpendicular to the interface





$$E_i \cos\theta_1 - E_r \cos\theta_1 = E_t \cos\theta_2$$

$$H_i + H_r = H_t$$

so $$\frac{1}{Z_1}(E_i + E_r) = \frac{1}{Z_2} E_t$$

$$r_{mp} = \frac{E_r}{E_i} = \frac{Z_1\cos\theta_1 - Z_2\cos\theta_2}{Z_1\cos\theta_1 + Z_2\cos\theta_2}$$

$$t_{mp} = \frac{E_t}{E_i} = \frac{2Z_2\cos\theta_1}{Z_1\cos\theta_1 + Z_2\cos\theta_2}$$

agnetic
arallel"

$$E_i + E_r = E_t$$

$$H_i \cos\theta_1 - H_r \cos\theta_1 = H_t \cos\theta_2$$

so $$\frac{\cos\theta_1}{Z_1}(E_i - E_r) = \frac{\cos\theta_2}{Z_2} E_t$$

$$r_{ep} = \frac{Z_2\cos\theta_1 - Z_1\cos\theta_2}{Z_2\cos\theta_1 + Z_1\cos\theta_2}$$

$$t_{ep} = \frac{2Z_2\cos\theta_1}{Z_2\cos\theta_1 + Z_1\cos\theta_2}$$

ep = "electric parallel"

"Fresnel equations" (θ_2 set i.t.o θ_1 by Snell's law)

$$\mu_1 = \mu_2 \qquad Z_1/Z_2 = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \sqrt{\frac{\mu\varepsilon_2}{\mu\varepsilon_1}} = \frac{v_1}{v_2} = \frac{n_2}{n_1}$$

When a monochromatic wave propagates from a high-index medium to a low index medium ($n_1 > n_2$) we can hit an interesting limit of Snell's law

$$\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1$$: if $\sin\theta_1 > \frac{n_2}{n_1}$, there is no solution for θ_2

when $\underbrace{\sin\theta_1}_{\substack{\parallel \\ \sin\theta_c}} = \frac{n_2}{n_1}$, $\theta_2 = \pi/2$ and the refracted wave moves along the boundary

critical angle

Consider the "mp" polarization from page 26 - in medium 2 the electric field is

$$\vec{E}_2(x,z,t) = \vec{E}_t\, e^{i(k_{2x}x + k_{2z}z - \omega t)}$$

$$k_{2x} = \frac{\omega}{c} n_2 \sin\theta_2 = \frac{\omega}{c} n_1 \sin\theta_1 \quad \text{by Snell's law}$$

$$k_{2z}^2 = \left(\frac{\omega}{c} n_2\right)^2 \cos^2\theta_2 = \left(\frac{\omega}{c} n_2\right)^2 \left(1 - \sin^2\theta_2\right) = \left(\frac{\omega}{c} n_2\right)^2 \left(1 - \frac{n_1^2}{n_2^2}\sin^2\theta_1\right)$$

$$= \left(\frac{\omega}{c}\right)^2 \left[n_2^2 - n_1^2 \sin^2\theta_1\right]$$

but if $\sin\theta_1 > n_2/n_1$ the object in square brackets is negative

and thus $k_{2z} = i\kappa \qquad \kappa = \frac{\omega}{c}\sqrt{n_1^2 \sin^2\theta_1 - n_2^2}$

$$\& \quad \vec{E}_2(x,z,t) = \vec{E}_t\, e^{-\kappa z}\, e^{i(\frac{\omega}{c} n_1 \sin\theta_1 x - \omega t)}$$

N.B. the square root chosen to get physically reasonable decaying exponential.

so we see that for $\theta_1 > \theta_c$, the wave propagates along the boundary, penetrating a distance $\sim 1/\kappa$ into medium 2.

notice that $$r_{mp} = \frac{\frac{\mu_1}{\mu_2}\frac{n_2}{n_1}\cos\theta_1 - \cos\theta_2}{\frac{\mu_1}{\mu_2}\frac{n_2}{n_1}\cos\theta_1 + \cos\theta_2} = \frac{\frac{\mu_1}{\mu_2}\frac{n_2}{n_1}\cos\theta_1 - i\frac{c}{\omega}\kappa}{\frac{\mu_1}{\mu_2}\frac{n_2}{n_1}\cos\theta_1 + i\frac{c}{\omega}\kappa} = \frac{x - iy}{x + iy} = e^{i\gamma}$$

so the reflected wave has the same amplitude as the incident wave, but is phase-shifted

"total internal reflection"

WAVES IN SIMPLE CONDUCTING MATTER

In "simple" conducting matter $\vec{D} = \varepsilon\vec{E}$, $\vec{B} = \mu\vec{H}$ and $\vec{J}_f = \sigma\vec{E}$ with ε, μ, σ constant.

With no free charge present the Maxwell equations become

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{H} = 0 \qquad \vec{\nabla}\times\vec{H} = \sigma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t}$$

$$-\nabla^2\vec{H} = \vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = -\sigma\mu\frac{\partial\vec{H}}{\partial t} - \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2}$$

$$-\nabla^2\vec{E} = \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\sigma\mu\frac{\partial\vec{H}}{\partial t} - \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2}$$

$$\left[\nabla^2 - \mu\sigma\frac{\partial}{\partial t} - \mu\varepsilon\frac{\partial^2}{\partial t^2}\right]\left\{\begin{matrix}\vec{E}\\ \vec{H}\end{matrix}\right\} = 0 \qquad [W]$$

this is a wave eqn with DAMPING. The presence of damping is to be expected as energy is constantly being lost from the wave through Joule heating

$$\frac{dW}{dt} = \int_V d^3\vec{r}\,\vec{J}_f\cdot\vec{E} = \sigma\int_V d^3\vec{r}\,|\vec{E}|^2$$

Let's consider whether monochromatic plane waves can propagate in our simple conductor:

$$\vec{E}(\vec{r},t) = \vec{E}\,e^{i(\vec{k}\cdot\vec{r}-\omega t)} \qquad \vec{H}(\vec{r},t) = \vec{H}\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\vec{\nabla}\cdot\vec{E} = 0 \rightarrow \vec{k}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t} \rightarrow \vec{k}\times\vec{E} = \omega\mu\vec{H}$$

$$\vec{\nabla}\cdot\vec{H} = 0 \rightarrow \vec{k}\cdot\vec{H} = 0 \qquad \vec{\nabla}\times\vec{H} = \sigma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t} \rightarrow \vec{k}\times\vec{H} = -\omega\left(\epsilon + i\sigma/\omega\right)\vec{E}$$

and we may interpret $\epsilon + i\sigma/\omega$ as a complex, frequency dependent, permittivity

$$\epsilon(\omega) = \epsilon_r + i\epsilon_i \qquad \epsilon_r = \varepsilon, \quad \epsilon_i = \sigma/\omega$$

$$[W] \rightarrow -k^2 + i\mu\omega\sigma + \mu\varepsilon\omega^2 = 0 \Rightarrow k^2 = (\mu\varepsilon)\omega^2 + (i\mu\sigma)\omega \equiv n^2(\omega)\,\omega^2/c^2$$

if we define a complex refractive index, $n(\omega)$

$\Rightarrow$ simple conductors exhibit "dispersion" (dependence on frequency)

$$k^2 = \vec{k}\cdot\vec{k} = \mu\varepsilon\omega^2 + i\mu\sigma\omega$$

so if we express $\vec{k}$ as a complex vector, $\vec{k} = \vec{q} + i\vec{\kappa}$, we have, taking the real and imaginary parts of the dispersion relation

$$q^2 - \kappa^2 = (\mu\varepsilon)\,\omega^2$$

$$\& \quad \vec{q}\cdot\vec{\kappa} = \tfrac{1}{2}\mu\sigma\omega$$

let's consider solutions where $\vec{q}$ & $\vec{\kappa}$ are parallel so $\vec{k} = k\hat{k}$ with $\hat{k}$ a real unit vector in the direction of propagation. We may then write

$$k\hat{k} = n(\omega)\frac{\omega}{c}\hat{k} = (n_r + i n_i)\frac{\omega}{c}\hat{k}$$

[sometimes a notation $n_r = n'$, $n_i = n''$ is used]

$$\& \quad n_r^2 - n_i^2 = \mu\varepsilon c^2 \quad \& \quad 2 n_r n_i = c^2\mu\sigma/\omega$$

which can be solved by

$$n_r = c\sqrt{\frac{\mu\varepsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]^{1/2}$$

$$n_i = c\sqrt{\frac{\mu\varepsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]^{1/2}$$

notice that $n_r \to c\sqrt{\mu\varepsilon} = c/v$, $n_i \to 0$ in the limit $\sigma \to 0$ as they should

for "good" conductors at "low" frequencies $\frac{\sigma}{\omega\varepsilon} \gg 1$ & $n_r \simeq n_i \simeq \sqrt{\frac{\sigma\mu}{2\omega}} = \frac{c/\omega}{\delta(\omega)} \gg 1$

with $\delta(\omega) = \sqrt{\frac{2}{\mu\sigma\omega}}$ as the "skin depth".

our solutions satisfy $\vec{k} = n(\omega)\frac{\omega}{c}\hat{k}$, $\hat{k}\cdot\vec{E} = 0$, $Z(\omega)\vec{H} = \hat{k}\times\vec{E}$

with $Z(\omega) = \sqrt{\frac{\mu}{\varepsilon(\omega)}} = \sqrt{\frac{\mu}{\varepsilon + i\sigma/\omega}}$

$$\vec{E}(\vec{r},t) = \vec{E}\, e^{-\frac{\omega}{c}n_i \hat{k}\cdot\vec{r}}\; e^{i(\frac{\omega}{c}n_r \hat{k}\cdot\vec{r} - \omega t)}$$

a damped plane wave moving in the $\hat{k}$ direction.