RADIATION

We're now ready to address the question of how we may generate wave-like e/m fields from charge & current distributions, and how these solutions can propagate over arbitrarily large distances.

Firstly recall that if we choose to work in LORENZ GAUGE, Maxwell's equations can be cast into the form

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t} \quad , \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

with the potentials being solutions of $\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\begin{Bmatrix}\varphi \\ \vec{A}\end{Bmatrix} = -\begin{Bmatrix}\rho/\varepsilon_0 \\ \mu_0\vec{J}\end{Bmatrix}$. [W]

Helpfully we've already found the general solution to the inhomogeneous wave eqn

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\psi(\vec{r},t) = f(\vec{r},t)$$

using Green's functions. We found that $\psi(\vec{r},t) = -\frac{1}{4\pi}\int d^3\vec{r}\,' \frac{1}{|\vec{r}-\vec{r}\,'|} f(\vec{r}\,', t - \frac{1}{c}|\vec{r}-\vec{r}\,'|)$

where we see that the field at time t is determined by the state of the source at time $t' = t - \frac{1}{c}|\vec{r}-\vec{r}\,'|$.

This "retardation" is associated with the finite speed, c, of wave propagation in vacuum.

Applying this solution to [W] we obtain

$$\boxed{\begin{aligned} \varphi(\vec{r},t) &= \frac{1}{4\pi\varepsilon_0}\int d^3\vec{r}\,' \frac{1}{|\vec{r}-\vec{r}\,'|}\rho(\vec{r}\,', t - \tfrac{1}{c}|\vec{r}-\vec{r}\,'|) \\ \& \quad \vec{A}(\vec{r},t) &= \frac{\mu_0}{4\pi}\int d^3\vec{r}\,' \frac{1}{|\vec{r}-\vec{r}\,'|}\vec{J}(\vec{r}\,', t - \tfrac{1}{c}|\vec{r}-\vec{r}\,'|) \end{aligned}}$$

an oscillating electric dipole

let's get an idea of what can happen by considering a (relatively) simple case – a point electric dipole at the origin whose dipole moment changes with time.

The charge density of the dipole is $\underline{\rho(\vec{r},t) = -\vec{p}(t)\cdot\vec{\nabla}\delta(\vec{r})}$ [see last semester]

and the corresponding current density follows from the continuity eqn $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$

$$\rho = -\vec{\nabla}\cdot\left[\vec{p}(t)\,\delta(\vec{r})\right]$$

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\left[\dot{\vec{p}}(t)\,\delta(\vec{r})\right] \qquad \Rightarrow \qquad \underline{\vec{J}(\vec{r},t) = \dot{\vec{p}}(t)\,\delta(\vec{r})}$$

The vector potential from this source is given by

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3\vec{r}\,'\,\frac{1}{|\vec{r}-\vec{r}\,'|}\,\delta(\vec{r}\,')\,\dot{\vec{p}}\left(t-\frac{1}{c}|\vec{r}-\vec{r}\,'|\right)$$

$$\underline{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\,\frac{1}{r}\,\dot{\vec{p}}(t-r/c)}$$

the easiest way to obtain φ is to use the Lorentz gauge condition $\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0$

$$\frac{\partial\varphi}{\partial t} = -c^2\,\vec{\nabla}\cdot\vec{A} = -\frac{\mu_0 c^2}{4\pi}\left[-\frac{\hat{r}}{r^2}\cdot\dot{\vec{p}} + \frac{1}{r}\,\vec{\nabla}\cdot\dot{\vec{p}}\right]$$

now it's useful to consider e.g. $\frac{\partial}{\partial x} f(t-r/c) = \frac{\partial}{\partial x} f\left(t-\frac{1}{c}\sqrt{x^2+y^2+z^2}\right)$

$$= -\frac{1}{c}\,\frac{x}{r}\,f'$$

so $\vec{\nabla}\cdot\dot{\vec{p}} = -\frac{1}{cr}\left(x\ddot{p}_x + y\ddot{p}_y + z\ddot{p}_z\right) = -\frac{1}{cr}\,\vec{r}\cdot\ddot{\vec{p}}$

& $\frac{\partial\varphi}{\partial t} = \frac{+1}{4\pi\varepsilon_0}\left[\frac{1}{r^2}\,\hat{r}\cdot\dot{\vec{p}} + \frac{1}{cr}\,\hat{r}\cdot\ddot{\vec{p}}\right] \Rightarrow \underline{\varphi(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\left[\frac{\hat{r}\cdot\dot{\vec{p}}(t-r/c)}{cr} + \frac{\hat{r}\cdot\vec{p}(t-r/c)}{r^2}\right]}$

[whenever I write $\vec{p}, \dot{\vec{p}}$ or $\ddot{\vec{p}}$ in a field at $(\vec{r},t)$, I mean it to be evaluated at the retarded time]

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{\mu_0}{4\pi}\vec{\nabla}\times\left[\frac{1}{r}\dot{\vec{p}}(t-r/c)\right] = \frac{\mu_0}{4\pi}\left[\vec{\nabla}\left(\frac{1}{r}\right)\times\dot{\vec{p}} + \frac{1}{r}\vec{\nabla}\times\dot{\vec{p}}\right]$$

$$\partial_j f_k(t-r/c) = -\frac{1}{cr} r_j f_k' \quad \Rightarrow \quad \epsilon_{ijk}\partial_j f_k = -\frac{1}{cr}\epsilon_{ijk} r_j f_k'$$

$$\& \ \vec{\nabla}\times\dot{\vec{p}} = -\frac{1}{cr}\vec{r}\times\ddot{\vec{p}}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{\mu_0}{4\pi}\left[-\frac{\vec{r}}{r^2}\times\dot{\vec{p}} - \frac{1}{cr^2}\vec{r}\times\ddot{\vec{p}}\right]$$

$$\boxed{\vec{B} = -\frac{\mu_0}{4\pi}\hat{r}\times\left[\frac{1}{r^2}\dot{\vec{p}} + \frac{1}{cr}\ddot{\vec{p}}\right]}$$

$$4\pi\epsilon_0\frac{\partial\vec{A}}{\partial t} = \mu_0\epsilon_0\frac{1}{r}\ddot{\vec{p}} = \frac{1}{c^2 r}\ddot{\vec{p}}$$

$$4\pi\epsilon_0\vec{\nabla}\varphi = \vec{\nabla}\left(\frac{\vec{r}\cdot\dot{\vec{p}}}{cr^2}\right) + \vec{\nabla}\left(\frac{\vec{r}\cdot\vec{p}}{r^3}\right) = -\frac{2}{cr^3}\hat{r}(\vec{r}\cdot\dot{\vec{p}}) + \frac{1}{cr^2}\vec{\nabla}(\vec{r}\cdot\dot{\vec{p}})$$
$$-\frac{3}{r^4}\hat{r}(\vec{r}\cdot\vec{p}) + \frac{1}{r^3}\vec{\nabla}(\vec{r}\cdot\vec{p})$$

now
$$\vec{\nabla}(\vec{r}\cdot\vec{a}) = (\vec{r}\cdot\vec{\nabla})\vec{a} + (\vec{a}\cdot\vec{\nabla})\vec{r} + \vec{r}\times(\vec{\nabla}\times\vec{a}) + \vec{a}\times(\vec{\nabla}\times\vec{r})$$
$$= (\vec{r}\cdot\vec{\nabla})\vec{a} + \vec{a} + \vec{r}\times(\vec{\nabla}\times\vec{a})$$

$$\vec{r}\cdot\vec{\nabla} f_k(t-r/c) = r_j\partial_j f_k = -\frac{1}{cr} r_j r_j f_k' = -\frac{r}{c}f_k'$$

so
$$4\pi\epsilon_0\vec{\nabla}\varphi = -\frac{2}{cr^2}\hat{r}(\hat{r}\cdot\dot{\vec{p}}) + \frac{1}{cr^2}\left[-\frac{r}{c}\ddot{\vec{p}} + \dot{\vec{p}} + \vec{r}\times(\vec{r}\times\ddot{\vec{p}})\left(-\frac{1}{cr}\right)\right]$$
$$-\frac{3}{r^3}\hat{r}(\hat{r}\cdot\vec{p}) + \frac{1}{r^3}\left[-\frac{r}{c}\dot{\vec{p}} + \vec{p} + \vec{r}\times(\vec{r}\times\dot{\vec{p}})\left(-\frac{1}{cr}\right)\right]$$

$$= -\frac{2}{cr^2}\hat{r}(\hat{r}\cdot\dot{\vec{p}}) - \frac{1}{c^2 r}\ddot{\vec{p}} - \frac{1}{c^2 r^3}\left(\vec{r}(\vec{r}\cdot\ddot{\vec{p}}) - \ddot{\vec{p}}r^2\right)$$
$$-\frac{3}{r^3}\hat{r}(\hat{r}\cdot\vec{p}) + \vec{p}/r^3 - \frac{1}{cr^4}\left(\vec{r}(\vec{r}\cdot\dot{\vec{p}}) - \dot{\vec{p}}r^2\right)$$

$$= -\frac{1}{c^2 r}\hat{r}(\hat{r}\cdot\ddot{\vec{p}}) - \frac{3}{cr^2}\hat{r}(\hat{r}\cdot\dot{\vec{p}}) + \frac{1}{cr^2}\dot{\vec{p}} - \frac{3}{r^3}\hat{r}(\hat{r}\cdot\vec{p}) + \frac{1}{r^3}\vec{p}$$

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0}\left[\frac{3\hat{r}(\hat{r}\cdot\vec{p}) - \vec{p}}{r^3} + \frac{3\hat{r}(\hat{r}\cdot\dot{\vec{p}}) - \dot{\vec{p}}}{cr^2} + \frac{\hat{r}(\hat{r}\cdot\ddot{\vec{p}}) - \ddot{\vec{p}}}{c^2 r}\right]}$$

$$\vec{B} = -\frac{\mu_0}{4\pi}\hat{r}\times\left[\frac{1}{r^2}\dot{\vec{p}} + \frac{1}{cr}\ddot{\vec{p}}\right]$$

- notice that at very short distances from the dipole, the field is indeed dipole-like,
- at intermediate distances, the fields depend upon $\vec{p}$, $\dot{\vec{p}}$ and $\ddot{\vec{p}}$, i.e. upon the charge positions, velocities and accelerations.
- at very large distances from the source only the $\frac{1}{r}$ terms will survive, and these are seen to be proportional to $\ddot{\vec{p}}$, the acceleration of charges.

It is the terms $\sim\frac{1}{r}$ which are describing RADIATION that propagates infinitely far from the source. We can get a better idea if we compute the flow of energy across the surface of a sphere of radius r centered on the dipole.

We want $\quad r^2\int d\Omega\ \hat{r}\cdot\vec{S} \quad$ where the Poynting vector $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$

For a dipole whose direction remains constant we obtain (after some algebra)

$$\frac{1}{4\pi\epsilon_0}\frac{2}{3}\left[\frac{d}{dt}\left(\frac{p^2}{2r^3} + \frac{p\dot{p}}{cr^2} + \frac{\dot{p}^2}{c^2 r}\right) + \frac{\ddot{p}^2}{c^3}\right]$$

Now if we average over a cycle of an oscillatory dipole, the total time derivative will contribute zero, but the final term $\quad \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{\ddot{p}^2}{c^3}$

is always positive & is independent of r, implying that energy is carried undiminished out to infinity.

This piece of the field is known as the RADIATION field.

more generally we can define whether a compact source RADIATES into a solid angle $d\Omega$ by computing

$$dP(t) = \lim_{r\to\infty}\left[\hat{r}\cdot\vec{S}(\vec{r},t)\, r^2 d\Omega\right]$$

if this quantity is non-zero, the source radiates. Note that this implies $\vec{S} \sim \hat{r}/r^2$ as $r\to\infty$ and $\vec{E}\sim 1/r$, $\vec{B}\sim 1/r$

The total power radiated to a sphere at infinity is

$$P(t) = \int_{\text{sph}\,\infty} d\vec{A}\cdot\vec{S}$$

but Poynting's theorem states $\int d\vec{A}\cdot\vec{S} + \int d^3\vec{r}\,\vec{J}\cdot\vec{E} = -\frac{dU_{em}}{dt}$ where U_{em} is the total energy stored in $\vec{E},\vec{B}$

thus $P(t) + \frac{dU_{em}}{dt} = -\int d^3\vec{r}\,\vec{J}\cdot\vec{E}$

& if the time-dependence is harmonic, $\vec{J}(\vec{r},t) = \vec{J}(\vec{r})e^{-i\omega t}$, then the time average of the total time derivative $\frac{dU_{em}}{dt}$, $\left\langle\frac{dU_{em}}{dt}\right\rangle = 0$ and

$$\langle P\rangle = -\frac{1}{2}\,\mathrm{Re}\int d^3\vec{r}\;\vec{J}\cdot\vec{E}^*$$

thus the work done to maintain the current source ends up as radiation energy far from the source

Radiation Fields

recall that in Lorenz gauge $\vec{A} = \frac{\mu_0}{4\pi}\int d^3\vec{r}'\, \frac{1}{|\vec{r}-\vec{r}'|}\, \vec{J}\left(\vec{r}', t - \frac{1}{c}|\vec{r}-\vec{r}'|\right)$

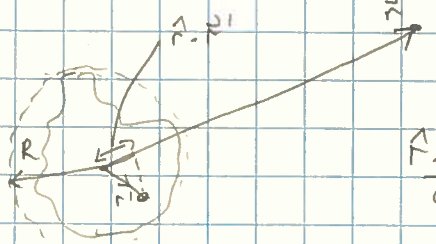
If the source is restricted to a finite region, having largest radius R, then the radiation field limit occurs for $r >> R$, and we may approximate

$$|\vec{r}-\vec{r}'| = r\sqrt{1 - 2\frac{\hat{r}.\vec{r}'}{r} + \left(\frac{r'}{r}\right)^2}$$

$$= r\left\{1 - \frac{1}{r}\hat{r}.\vec{r}' + \frac{1}{2}\left(\frac{r'}{r}\right)^2\left[1-(\hat{r}.\hat{r}')^2\right] + \dots\right\}$$

$$= r - \hat{r}.\vec{r}' + \frac{r'^2}{2r}\left[1-(\hat{r}.\hat{r}')^2\right] + \dots$$

We'll retain the term $\hat{r}.\vec{r}'/c$ in the retarded time - we can see that it accounts for the time needed for an electromagnetic signal to exit the source



$$\frac{\hat{r}.\vec{r}'}{c} \le \frac{R}{c}$$

Thus $\vec{A}_{rad}(\vec{r},t) = \frac{\mu_0}{4\pi r}\int d^3\vec{r}'\, \vec{J}\left(\vec{r}', t - \frac{r}{c} + \frac{\hat{r}.\vec{r}'}{c}\right)$

& similarly $\varphi_{rad}(\vec{r},t) = \frac{\mu_0 c^2}{4\pi}\frac{1}{r}\int d^3\vec{r}'\, \rho\left(\vec{r}', t - \frac{r}{c} + \frac{\hat{r}.\vec{r}'}{c}\right)$

The easiest way to analyse these equations & to obtain $\vec{E}_{rad}, \vec{B}_{rad}$ is to move to the frequency domain by Fourier transforming

$$\vec{A}_{rad}(\vec{r},t) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\ \vec{A}_{rad}(\vec{r},\omega)\, e^{-i\omega t}$$

d $$\vec{J}(\vec{r},t) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\ \vec{J}(\vec{r},\omega) e^{-i\omega t}$$

o $$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\ \vec{A}_{rad}(\vec{r},\omega) e^{-i\omega t} = \frac{\mu_0}{4\pi}\frac{1}{r}\int d^3\vec{r}\,' \ \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\ \vec{J}(\vec{r}\,',\omega)\, e^{-i\omega\left(t - r/c + \frac{\hat{r}\cdot\vec{r}\,'}{c}\right)}$$

$$\Rightarrow \quad \vec{A}_{rad}(\vec{r},\omega) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3\vec{r}\,' \ \vec{J}(\vec{r}\,',\omega)\, e^{-i\vec{k}\cdot\vec{r}\,'} \qquad \text{if } \vec{k} \equiv \frac{\omega}{c}\hat{r}$$

$$\vec{A}_{rad}(\vec{r},\omega) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\underbrace{\vec{J}(\vec{k},\omega)}$$

the full space-time fourier transform of the current density

similarly $$\varphi_{rad}(\vec{r},\omega) = \frac{\mu_0 c^2}{4\pi}\frac{e^{ikr}}{r}\rho(\vec{k},\omega)$$

$$\vec{B} = \vec{\nabla}\times\vec{A} \quad \Rightarrow \quad \vec{B}_{rad}(\vec{r},\omega) = \frac{\mu_0}{4\pi}\vec{\nabla}\times\left(\frac{e^{ikr}}{r}\vec{J}(\vec{k},\omega)\right)$$

$$= \frac{\mu_0}{4\pi}\vec{\nabla}\left(\frac{e^{ikr}}{r}\right)\times\vec{J}(\vec{k},\omega)$$

[* we should have a term $\vec{\nabla}\times\vec{J}$ here
– check that it's not needed in the radiation region]

$$= \frac{\mu_0}{4\pi}\left(ik\frac{e^{ikr}}{r} - \frac{e^{ikr}}{r^2}\right)\hat{r}\times\vec{J}(\vec{k},\omega)$$

but the $1/r^2$ term is negligable in the radiation region, so

$$\boxed{\vec{B}_{rad}(\vec{r},\omega) = i\frac{\omega}{c}\hat{r}\times\vec{A}_{rad}(\vec{r},\omega)}$$

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\varphi + i\omega\vec{A}$$

$$\vec{E}_{rad} = \frac{\mu_0 c^2}{4\pi}(-ik\hat{r})\frac{e^{ikr}}{r}\rho(\vec{k},\omega) + i\omega\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\vec{J}(\vec{k},\omega)$$

$$= i\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left[-c\omega\hat{r}\rho(\vec{k},\omega) + \omega\vec{J}(\vec{k},\omega)\right]$$

the continuity equation $\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$ $\Rightarrow$ $i\vec{k}\cdot\vec{J} = i\omega\rho$

$\Rightarrow$ $c\omega\rho = \omega\hat{r}\cdot\vec{J}$

$$\vec{E}_{rad} = \frac{i\omega\mu_0}{4\pi}\frac{e^{ikr}}{r}\left[\underbrace{-\hat{r}(\hat{r}\cdot\vec{J}) + \vec{J}}_{-\hat{r}\times(\hat{r}\times\vec{J})}\right]$$

$$= -i\omega\,\hat{r}\times\left(\hat{r}\times\left[\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\vec{J}\right]\right)$$

$$\boxed{\vec{E}_{rad}(\vec{r},\omega) = -i\omega\,\hat{r}\times(\hat{r}\times\vec{A}_{rad}(\vec{r},\omega))}$$

$\vec{A}_{rad}(\vec{r},\omega) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\vec{J}(\vec{k},\omega)$ | $\vec{B}_{rad}(\vec{r},\omega) = i\frac{\omega}{c}\hat{r}\times\vec{A}_{rad}$ | $\vec{E}_{rad}(\vec{r},\omega) = -i\omega\,\hat{r}\times(\hat{r}\times\vec{A}_{rad})$

N.B all fall like $1/r$ at large distances.

notice that $\vec{E}_{rad} = -c\,\hat{r}\times\vec{B}_{rad}$ and $\vec{E}_{rad}$ is perpendicular to both $\hat{r}$ and $\vec{B}_{rad}$.

The time averaged flux of energy $\langle\vec{S}\rangle = \frac{1}{2}\frac{1}{\mu_0}\mathrm{Re}\left(\vec{E}^*\times\vec{B}\right) = \frac{c}{2\mu_0}|\vec{B}|^2$

$$\langle\vec{S}\rangle = \frac{\omega^2}{2\mu_0 c}|\hat{r}\times\vec{A}_{rad}|^2 = \frac{\mu_0\omega^2}{32\pi^2 c}\frac{1}{r^2}|\hat{r}\times\vec{J}|^2$$

which indicates that only the current perpendicular to $\hat{r}$ contributes to radiation.

Larmor's formula - radiation from a slowly-moving accelerating charge

a single point charge q whose position is $\vec{r}_0(t)$ causes a current density

$$\vec{J}(\vec{r}',t') = q\,\dot{\vec{r}}_0(t')\,\delta(\vec{r}'-\vec{r}_0(t'))$$

$$\vec{A}_{rad}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{1}{r}\int d^3\vec{r}'\;\vec{J}(\vec{r}', t-r/c + \hat{r}\cdot\vec{r}'/c)$$

for a slow moving particle, $v \ll c$, we can neglect the term $\frac{\hat{r}\cdot\vec{r}'}{c}$

(e.g. if particle at $r=0$ at $t=0$
$\vec{r}'(t) = \vec{v}(0)t + \ldots$
$t + \frac{\hat{r}\cdot\vec{r}'}{c} \simeq t(1+v_r/c) \simeq t$

$$\vec{A}_{rad}(\vec{r},t) \simeq \frac{\mu_0}{4\pi}\frac{q}{r}\int d^3\vec{r}'\;\dot{\vec{r}}_0(t-r/c)\,\delta(\vec{r}'-\vec{r}_0(t-r/c))$$

$$\simeq \frac{\mu_0}{4\pi}\frac{q}{r}\dot{\vec{r}}_0(t-r/c)$$

from $\vec{E}_{rad}(\vec{r},\omega) = -i\omega\,\hat{r}\times(\hat{r}\times\vec{A}_{rad}(\vec{r},\omega))$ we'd expect $\vec{E}_{rad}(\vec{r},t) = \hat{r}\times\left(\hat{r}\times\frac{\partial}{\partial t}\vec{A}_{rad}(\vec{r},t)\right)$

and thus

$$\vec{E}_{rad}(\vec{r},t) \simeq \frac{\mu_0}{4\pi}\frac{q}{r}\hat{r}\times(\hat{r}\times\vec{a}(t-r/c))$$

$$= -\frac{\mu_0}{4\pi}\frac{q}{r}\left(\vec{a}(t-r/c) - \hat{r}(\hat{r}\cdot\vec{a}(t-r/c))\right)$$

$$= -\frac{\mu_0}{4\pi}\frac{q}{r}\vec{a}_\perp(t-r/c)$$

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0 c}\vec{E}\times(\hat{r}\times\vec{E}) = \frac{1}{\mu_0 c}\hat{r}\,|\vec{E}|^2$$

& the power radiated into a solid angle $d\Omega$ is $\frac{dP}{d\Omega} = r^2\cdot\frac{1}{\mu_0 c}\frac{\mu_0^2}{(4\pi)^2}\frac{q^2}{r^2}\left(\vec{a}^2 - (\hat{r}\cdot\vec{a})^2\right)$

if $\hat{r}\cdot\vec{a} = a\cos\theta$ $\quad \frac{dP}{d\Omega} = \frac{\mu_0 q^2}{16\pi^2 c}|\vec{a}(t-r/c)|^2\sin^2\theta$ $\quad$ & $P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2}{c^3}|\vec{a}_{ret}|^2$

(CARTESIAN) MULTIPOLE EXPANSION OF RADIATION

As in the static case it's useful to consider an expansion of the radiation field in terms of multipoles. For sources of small size or slow time variation the field is often well described by only the first few terms in the expansion.

We previously obtained (for monochromatic fields)
$$\vec{B}_{rad}^{(\vec{r},\omega)} = \frac{i}{c}\,\omega\,\hat{r}\times\vec{A}_{rad} \qquad \vec{E}_{rad}^{(\vec{r},\omega)} = -i\omega\,\hat{r}\times(\hat{r}\times\vec{A}_{rad})$$

which generalizes to
$$\vec{B}(\vec{r},t) = -\frac{1}{c}\,\hat{r}\times\frac{\partial\vec{A}_{rad}}{\partial t} \qquad \vec{E}_{rad}(\vec{r},t) = \hat{r}\times\left(\hat{r}\times\frac{\partial\vec{A}_{rad}}{\partial t}\right)$$

so it appears that $\dfrac{\partial\vec{A}_{rad}}{\partial t}$ is the quantity that controls radiation.

Let's define the RADIATION VECTOR, $\vec{\alpha}(\vec{r},t)$, by $\dfrac{\partial\vec{A}_{rad}}{\partial t} = \dfrac{\mu_0}{4\pi r}\,\vec{\alpha}(\vec{r},t)$

$$\text{so that} \quad \vec{\alpha}(\vec{r},t) = \frac{\partial}{\partial t}\int d^3\vec{r}\,'\;\vec{J}\left(\vec{r}\,',\, t-r/c+\frac{1}{c}\hat{r}\cdot\vec{r}\,'\right)$$

The angular distribution of radiated power is $\dfrac{dP}{d\Omega} = \dfrac{\mu_0}{16\pi^2 c}\left|\hat{r}\times\vec{\alpha}(\vec{r},t)\right|^2$

Recall that previously we argued that $\frac{1}{c}\hat{r}\cdot\vec{r}\,' \le L/c$ for a source of size L is the time taken for an e/m signal to travel accross the source. Comparing this time to $t-r/c$ we see that for a physically small source, or for slow time variation of the fields, an expansion in small $\frac{1}{c}\hat{r}\cdot\vec{r}\,'$ compared with $t-r/c$ is well motivated.

$$\vec{J}\left(\vec{r}\,',\, t-r/c+\frac{1}{c}\hat{r}\cdot\vec{r}\,'\right) = \vec{J}(\vec{r}\,', t-r/c) + \frac{1}{c}\hat{r}\cdot\vec{r}\,'\frac{\partial}{\partial t}\vec{J}(\vec{r}\,', t-r/c)$$
$$+\frac{1}{2}\left(\frac{1}{c}\hat{r}\cdot\vec{r}\,'\right)^2\frac{\partial^2}{\partial t^2}\vec{J}(\vec{r}\,', t-r/c) + \ldots \qquad [J]$$

e.g. for monochromatic current variation, $\vec{J}\sim e^{-i\omega t}$, the expansion is in powers of $\dfrac{\omega L}{c}$ and since $\omega = ck = \dfrac{2\pi c}{\lambda}$, the small parameter is $\dfrac{L}{\lambda}$,

and the expansion will work for sources much smaller than the wavelength

We will show that the corresponding expansion of the radiation vector is

$$\vec{\alpha}(\vec{r},t) = \frac{d^2}{dt^2}\Big[\underbrace{\vec{p}(t-r/c)}_{\text{electric dipole moment}}\Big] + \frac{1}{c}\frac{d^2}{dt^2}\Big[\underbrace{\vec{m}(t-r/c)}_{\text{magnetic dipole moment}}\times\hat{r}\Big] + \frac{1}{c}\frac{d^3}{dt^3}\Big[\underbrace{\overleftrightarrow{Q}(t-r/c)}_{\text{electric quadrupole tensor}}\cdot\hat{r}\Big] + \dots$$

$\boxed{\text{electric dipole radiation}}$ "E1"

retaining only the first term in the expansion [J] we have

$$\vec{\alpha}_{E1}(\vec{r},t) = \frac{d}{dt}\int d^3\vec{r}\,'\,\vec{J}(\vec{r}\,',t-r/c)$$

$$= \frac{d^2}{dt^2}\vec{p}(t-r/c) \equiv \ddot{\vec{p}}_{ret}$$

[We can relate this to the charge density:

$$\int d^3\vec{r}\;\vec{\nabla}\cdot(r_i\vec{J}) = \int d\vec{S}\cdot\vec{J}\,r_i = 0 \quad \text{for localized } \vec{J}$$

$$\| $$

$$\int d^3\vec{r}\;\partial_j(r_iJ_j) = \int d^3\vec{r}\,J_i + \int d^3\vec{r}\;r_i\vec{\nabla}\cdot\vec{J}$$

$$\Rightarrow 0 = \int d^3\vec{r}\,\vec{J} - \int d^3\vec{r}\;\vec{r}\frac{\partial\rho}{\partial t}$$

$$\Rightarrow \int d^3\vec{r}\,\vec{J} = \frac{\partial}{\partial t}\int d^3\vec{r}\;\vec{r}\rho = \frac{\partial}{\partial t}\vec{p}$$
]

& thus $$\vec{B}_{E1} = -\frac{\mu_0}{4\pi c}\frac{\hat{r}\times\ddot{\vec{p}}_{ret}}{r}$$

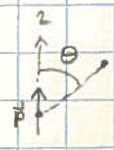
$$\vec{E}_{E1} = \frac{\mu_0}{4\pi}\frac{\hat{r}(\hat{r}\cdot\ddot{\vec{p}}_{ret}) - \ddot{\vec{p}}_{ret}}{r}$$

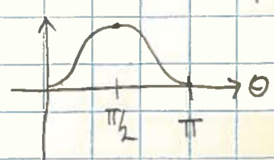
which are indeed the long-distance terms we obtained in the field from a time-dependant electric dipole.

the angular distribution of power $$\left(\frac{dP}{d\Omega}\right)_{E1} = \frac{\mu_0}{16\pi^2 c}\left|\hat{r}\times\ddot{\vec{p}}_{ret}\right|^2$$

e.g. if we define the z-axis to be along the direction of $\vec{p}$ (assuming $\vec{p}$ is not rotating)
then $|\hat{r}\times\vec{p}| \sim |\hat{r}\times\hat{z}| \sim \sin\theta$

$$\left(\frac{dP}{d\Omega}\right)_{E1} = \frac{\mu_0}{16\pi^2 c}\left|\ddot{\vec{p}}_{ret}\right|^2\sin^2\theta$$





$$P_{E1} = \frac{\mu_0}{6\pi c}\left|\ddot{\vec{p}}_{ret}\right|^2$$

for a time-harmonic electric dipole moment, $\vec{p}(t) = \vec{p}\, e^{-i\omega t}$, $\ddot{\vec{p}} = -\omega^2 \vec{p}$, and

$$\vec{B}_{E1} = \frac{\mu_0}{4\pi} \frac{\omega^2}{c^2} \frac{\hat{r}\times\vec{p}}{r} e^{-i\omega(t - r/c)} = \frac{\mu_0}{4\pi} \frac{\omega^2}{c^2} (\hat{r}\times\vec{p}) \frac{e^{i(kr-\omega t)}}{r}$$

$$\& \; \vec{E}_{E1} = -\frac{\mu_0}{4\pi} \omega^2 \left[\hat{r}\times(\hat{r}\times\vec{p})\right] \frac{e^{i(kr-\omega t)}}{r}$$

[remember to take the real part]

the time-averaged angular power distribution is $\left\langle \left(\frac{dP}{d\Omega}\right)_{E1} \right\rangle = \frac{\mu_0 \omega^4}{16\pi^2 c} \frac{1}{2} (\hat{r}\times\vec{p})\cdot(\hat{r}\times\vec{p}^*)$

and the total power radiated over all angles $\langle P_{E1} \rangle = \frac{\mu_0 \omega^4}{12\pi c} \vec{p}\cdot\vec{p}^*$

magnetic dipole radiation "M1"

If a system does not have a suitable electric dipole moment, the next largest contribution comes from its magnetic dipole moment. Consider the next term in the expansion [J]

$$\frac{1}{c}\hat{r}\cdot\vec{r}\,'\frac{\partial}{\partial t}\vec{J}(\vec{r}\,', t-r/c)$$

which features the object $\hat{r}\cdot\vec{r}\,'\vec{J}$

Now since $\hat{r}\times(\vec{r}\,'\times\vec{J}) = \vec{r}\,'(\vec{J}\cdot\hat{r}) - (\hat{r}\cdot\vec{r}\,')\vec{J}$

we can rewrite $\hat{r}\cdot\vec{r}\,'\vec{J} = (\vec{J}\cdot\hat{r})\vec{r}\,' - \hat{r}\times(\vec{r}\,'\times\vec{J})$

and this can be reexpressed as a sum of two terms, one symmetric in $\vec{J}\leftrightarrow\vec{r}\,'$ and one antisymmetric

$$(\hat{r}\cdot\vec{r}\,')\vec{J} = \frac{1}{2}(\hat{r}\cdot\vec{r}\,')\vec{J} + \frac{1}{2}(\hat{r}\cdot\vec{r}\,')\vec{J}$$

$$= \frac{1}{2}(\hat{r}\cdot\vec{r}\,')\vec{J} + \frac{1}{2}(\hat{r}\cdot\vec{J})\vec{r}\,' + \frac{1}{2}(\vec{r}\,'\times\vec{J})\times\hat{r}$$

$$= \underbrace{\frac{1}{2}\left[(\hat{r}\cdot\vec{r}\,')\vec{J} + (\hat{r}\cdot\vec{J})\vec{r}\,'\right]}_{S} + \underbrace{\frac{1}{2}(\vec{r}\,'\times\vec{J})\times\hat{r}}_{AS}$$

S ↳ electric quadrupole "E2" (we'll come back to this later)

AS ↳ magnetic dipole "M1"

$$\vec{m} = \frac{1}{2}\int d^3\vec{r}\,'\;\vec{r}\,'\times\vec{J}$$

(see last semester)

$$\Rightarrow \vec{\alpha}_{M1}(\vec{r},t) = \frac{1}{c}\ddot{\vec{m}}(t-r/c)\times\hat{r}$$

$$\rightarrow \vec{B}_{M1} = \frac{-\mu_0}{4\pi c}\frac{1}{r}\hat{r}\times\vec{\alpha}_{M1} = \frac{\mu_0}{4\pi c^2}\frac{\hat{r}(\hat{r}\cdot\ddot{\vec{m}}_{ret}) - \ddot{\vec{m}}_{ret}}{r}$$

$$\rightarrow \vec{E}_{M1} = \frac{\mu_0}{4\pi c}\frac{\hat{r}\times\ddot{\vec{m}}_{ret}}{r}$$

$$\left(\frac{dP}{d\Omega}\right)_{M1} = \frac{\mu_0}{16\pi^2 c^3}\left|\hat{r}\times\ddot{\vec{m}}_{ret}\right|^2$$

$\boxed{\text{electric quadrupole radiation}}$ "E2"

$$\vec{\alpha}_{E2}(\vec{r},t) = \frac{1}{2}\frac{1}{c}\frac{d^2}{dt^2}\int d^3\vec{r}\,'\left[(\hat{r}\cdot\vec{r}\,')\vec{J} + (\hat{r}\cdot\vec{J})\vec{r}\,'\right]$$

which features $\hat{r}_i \int d^3\vec{r}\,'(r_i' J_j + J_i r_j') = \hat{r}_i X_{ij}$

$$\int d^3\vec{r}\;\vec{\nabla}\cdot(r_i r_j \vec{J}) = \int d\vec{S}\cdot\vec{J}\, r_i r_j = 0 \text{ for a localized } \vec{J}$$

$$= \int d^3\vec{r}\;\partial_k(r_i r_j J_k) = \int d^3\vec{r}\left(r_j J_i + r_i J_j + r_i r_j \underbrace{\vec{\nabla}\cdot\vec{J}}_{\to -\frac{\partial\rho}{\partial t}}\right)$$

$$0 = X_{ij} - \frac{\partial}{\partial t}\int d^3\vec{r}\,'\rho(\vec{r}\,';t)\,r_i' r_j'$$

$$X_{ij} = 2\frac{\partial}{\partial t}\frac{1}{2}\int d^3\vec{r}\,'\rho(\vec{r}\,';t)r_i' r_j' = 2\frac{\partial}{\partial t}Q_{ij}$$

So $$\vec{\alpha}_{E2}(\vec{r},t) = \frac{1}{c}\frac{d^3}{dt^3}\overleftrightarrow{Q}(t-r/c)\cdot\hat{r} = \frac{1}{c}\dddot{\overleftrightarrow{Q}}_{ret}\cdot\hat{r}$$

$$\vec{B}_{E2} = \frac{-\mu_0}{4\pi c^2}\frac{\hat{r}\times(\dddot{Q}\cdot\hat{r})}{r} \quad \text{or} \quad B^i_{E2} = \frac{-\mu_0}{4\pi c^2}\frac{1}{r}\epsilon_{ijk}\hat{r}_j\dddot{Q}_{kl}\hat{r}_l$$

$$\vec{E}_{E2} = \frac{\mu_0}{4\pi c}\frac{(\hat{r}\cdot\dddot{Q}\cdot\hat{r})\hat{r} - \dddot{Q}\cdot\hat{r}}{r} \quad \text{or} \quad E^i_{E2} = \frac{\mu_0}{4\pi c}\frac{1}{r}\left[\hat{r}_i(\hat{r}_j\dddot{Q}_{jk}\hat{r}_k) - \dddot{Q}_{ij}\hat{r}_j\right]$$

$$\left(\frac{dP}{d\Omega}\right)_{E2} = \frac{\mu_0}{16\pi^2 c^3}\left|\hat{r}\times(\dddot{Q}\cdot\hat{r})\right|^2$$

$$|\hat{r}\times(\dddot{Q}\cdot\hat{r})|^2 = \epsilon_{ijk}\hat{r}_j\dddot{Q}_{kl}\hat{r}_l\,\epsilon_{imn}\hat{r}_m\dddot{Q}_{np}\hat{r}_p = \epsilon_{ijk}\epsilon_{imn}\hat{r}_j\hat{r}_l\hat{r}_m\hat{r}_p\dddot{Q}_{kl}\dddot{Q}_{np} = (\delta_{jm}\delta_{kn}-\delta_{jn}\delta_{km})\hat{r}_j\hat{r}_l\hat{r}_m\hat{r}_p\dddot{Q}_{kl}\dddot{Q}_{np}$$

$$= \hat{r}_l\hat{r}_p\dddot{Q}_{nl}\dddot{Q}_{np} - \hat{r}_n\hat{r}_l\hat{r}_k\hat{r}_p\dddot{Q}_{kl}\dddot{Q}_{np}$$

$$\int d\Omega\,\hat{r}_l\hat{r}_p = \int_0^{2\pi}d\phi\int_{-1}^{1}d(\cos\theta)\left[\hat{x}_l\sin\theta\cos\phi + \hat{y}_l\sin\theta\sin\phi + \hat{z}_l\cos\theta\right]\left[\hat{x}_p\sin\theta\cos\phi + \hat{y}_p\sin\theta\sin\phi + \hat{z}_p\cos\theta\right]$$

all cross-terms integrate to zero over ϕ

$$= \int d\phi\int d(\cos\theta)\left[\hat{x}_l\hat{x}_p\sin^2\theta\cos^2\phi + \hat{y}_l\hat{y}_p\sin^2\theta\sin^2\phi + \hat{z}_l\hat{z}_p\cos^2\theta\right] = \frac{4\pi}{3}\left[\hat{x}_l\hat{x}_p + \hat{y}_l\hat{y}_p + \hat{z}_l\hat{z}_p\right]$$

$$\int d\Omega \, \hat{r}_j \hat{r}_\ell \hat{r}_m \hat{r}_p = ?$$

answer must be symmetric under exchange of any pair of indices

$$= N \left[\delta_{j\ell} \delta_{mp} + \delta_{jm} \delta_{\ell p} + \delta_{jp} \delta_{\ell m} \right]$$

e.g. $j = \ell = m = p = z$

$$\Rightarrow \int d\Omega \cos^4\theta = N \left[1+1+1 \right] = 3N$$

$$2\pi \int_{-1}^{1} dx \, x^4 = 2\pi \frac{2}{5} = \frac{4\pi}{5} \quad \Rightarrow N = 4\pi/15$$

$$\int d\Omega \, \hat{r}_j \hat{r}_\ell \hat{r}_m \hat{r}_p = \frac{4\pi}{15} \left[\delta_{j\ell} \delta_{mp} + \delta_{jm} \delta_{\ell p} + \delta_{jp} \delta_{\ell m} \right]$$

$$P_{E2} = \int d\Omega \left(\frac{dP}{d\Omega} \right)_{E2} = \frac{\mu_0}{16\pi^2 c^3} \cdot \left[\frac{4\pi}{3} \delta_{\ell p} \dddot{Q}_{n\ell} \dddot{Q}_{np} - \frac{4\pi}{15} \dddot{Q}_{m\ell} \dddot{Q}_{jp} \left(\delta_{j\ell}\delta_{mp} + \delta_{jm}\delta_{\ell p} + \delta_{jp}\delta_{\ell m} \right) \right]$$

$$= \frac{\mu_0}{16\pi^2 c^3} \left[\frac{4\pi}{3} \dddot{Q}_{n\ell} \dddot{Q}_{n\ell} - \frac{4\pi}{15} \left(\dddot{Q}_{pj} \dddot{Q}_{jp} + \dddot{Q}_{j\ell} \dddot{Q}_{j\ell} + \dddot{Q}_{\ell\ell} \dddot{Q}_{jj} \right) \right]$$

$$= \frac{\mu_0}{16\pi^2 c^3} \frac{4\pi}{15} \left[\dddot{Q}_{n\ell} \dddot{Q}_{n\ell} (5-1-1) - \dddot{Q}_{\ell\ell} \dddot{Q}_{nn} \right]$$

$$\underline{P_{E2} = \frac{\mu_0}{20\pi c^3} \left[\dddot{Q}_{n\ell} \dddot{Q}_{n\ell} - \frac{1}{3} \dddot{Q}_{\ell\ell} \dddot{Q}_{nn} \right]_{ret}}$$

or if we define the TRACELESS cartesian electric quadrupole tensor

$$\Theta_{ij} \equiv \frac{1}{2} \int d^3\vec{r}' \rho(\vec{r}') (3 r_i r_j - r^2 \delta_{ij}) = 3Q_{ij} - Q_{kk}\delta_{ij}$$

then $$P_{E2} = \frac{\mu_0}{20\pi c^3} \left[\left(\frac{1}{3} \dddot{\Theta}_{n\ell} + \frac{1}{3} \dddot{Q}_{kk} \delta_{n\ell} \right) \left(\frac{1}{3} \dddot{\Theta}_{n\ell} + \frac{1}{3} \dddot{Q}_{jj} \delta_{n\ell} \right) - \frac{1}{3} \dddot{Q}_{\ell\ell} \dddot{Q}_{nn} \right]$$

$$= \frac{\mu_0}{20\pi c^3} \left[\frac{1}{9} \dddot{\Theta}_{n\ell} \dddot{\Theta}_{n\ell} + \frac{1}{9} \dddot{Q}_{kk} \dddot{Q}_{jj} \delta_{nn} - \frac{1}{3} \dddot{Q}_{\ell\ell} \dddot{Q}_{nn} \right]$$

$$\underline{P_{E2} = \frac{\mu_0}{180\pi c^3} \left[\dddot{\Theta}_{n\ell} \dddot{\Theta}_{n\ell} \right]_{ret}}$$

relative magnitudes of E1, M1, E2 radiation (dimensional estimates)

$$\frac{P_{M1}}{P_{E1}} \sim \left(\frac{m/c}{p}\right)^2$$

for a charge distribution in motion, $\vec{J} = \rho\vec{v}$

$$\vec{m} = \frac{1}{2}\int d^3\vec{r}\;\vec{r}\times\vec{J}$$

$$\vec{p} = \int d^3\vec{r}\;\vec{r}\rho$$

$$\frac{m}{p} \sim \frac{rJ}{r\rho} \sim v$$

$$P_{M1}/P_{E1} \sim \left(v/c\right)^2$$ & for slow motions, $E1 \gg M1$

for harmonic time dependence $(\rho, \vec{J}) \sim e^{-i\omega t}$

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t} \Rightarrow J/L \sim \omega\rho$$ (L = characteristic size of distn)

then $$\frac{m}{p} \sim \frac{\omega L\rho}{c\rho} \sim kL \sim \frac{L}{\lambda}$$

$$\Rightarrow P_{M1}/P_{E1} \sim \left(L/\lambda\right)^2$$ & for small distn (compared to wavelength) $E1 \gg M1$

$$\frac{P_{E2}}{P_{E1}} \sim \left(\frac{\dddot{\Theta}/c}{\ddot{p}}\right)^2$$

for harmonic time dependence $\dddot{\Theta} \sim \omega^3\Theta \sim \omega^3 L^2\rho$

$$\ddot{p} \sim \omega^2 p \sim \omega^2 L\rho$$

$$P_{E2}/P_{E1} \sim \left(\frac{\omega L}{c}\right)^2 \sim \left(\frac{L}{\lambda}\right)^2$$

the relative size of E2 to M1 is a more subtle issue, but for distributions having certain symmetries, only one or the other contributes.

to go beyond E2 to higher multipoles, it is more convenient to use a spherical basis – we will not consider this here, it is explained in Jackson or Zangwill.