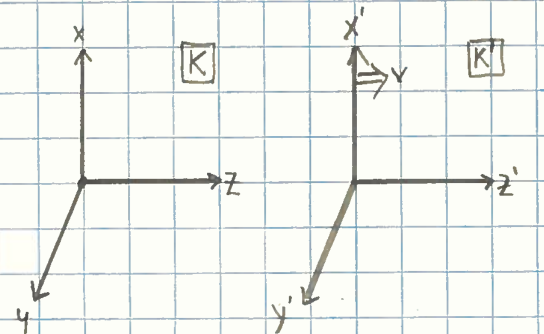
SPECIAL RELATIVITY

So far we've mainly considered how charge and current distributions produce electric and magnetic fields as measured by an observer in one particular reference frame. But we might wonder what an observer in a different reference frame, say one moving at a constant velocity relative to the first, would measure.

We'd guess that something interesting might result if we think about electromagnetic induction. Suppose we have a bar magnet which moves away from a loop of wire at a constant velocity. From the viewpoint of the stationary loop of wire, the moving magnet produces an electric field that induces current to flow in the loop. However in a frame moving along with the magnet, the stationary magnet produces no electric field, only a magnetic field - this magnetic field produces a force on the charges in the (moving) loop that induces the current.

It turns out that we can reconcile these two apparently different mechanisms, but in doing so we will abandon a "self-evident" truth assumed in e.g. Newton's mechanics - GALILEAN RELATIVITY.

Consider two frames K and K' ~ K' moves at a constant velocity of v in the z-direction. At time $t = t' = 0$, the two frames coincide.



The GALILEAN transformation relates positions & times in the two frames: $\vec{r}\,' = \vec{r} - \vec{v}t$

$t' = t$

thus $\dfrac{d\vec{r}\,'}{dt'} = \dfrac{d\vec{r}\,'}{dt} = \dfrac{d\vec{r}}{dt} - \vec{v}$ so $\vec{u}\,' = \vec{u} - \vec{v}$, $\vec{u} = \vec{u}\,' + \vec{v}$ and the velocity of an object in K is the velocity in K' plus the velocity of K.

Newton's second law is invariant under this transformation $\vec{F} = m\dfrac{d^2\vec{r}}{dt^2}$

$\dfrac{d^2\vec{r}\,'}{dt'^2} = \dfrac{d^2\vec{r}}{dt^2}$ so Galilean relativity has it that a force measured in K is the same as the force measured in K'.

Let's quickly explore what Galilean relativity implies for solutions of the wave equation. Restricting to the one-dimensional case for simplicity, we have

$$\left[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] f(z,t) = 0 \quad [W]$$

and the Galilean transformations are $(z,t) \to (z',t')$ with $z' = z - vt$, $t' = t$
so

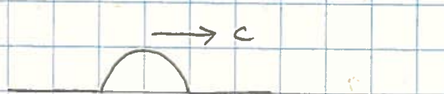
$$\left.\frac{\partial}{\partial z}\right)_t = \left.\frac{\partial z'}{\partial z}\right)_t \frac{\partial}{\partial z'} + \left.\frac{\partial t'}{\partial z}\right)_t \frac{\partial}{\partial t'} = \frac{\partial}{\partial z'}$$

$$\left.\frac{\partial}{\partial t}\right)_z = \left.\frac{\partial z'}{\partial t}\right)_z \frac{\partial}{\partial z'} + \left.\frac{\partial t'}{\partial t}\right)_z \frac{\partial}{\partial t'} = -v\frac{\partial}{\partial z'} + \frac{\partial}{\partial t'}$$

& the wave equation becomes $\left[\left(1 - v^2/c^2\right)\frac{\partial^2}{\partial z'^2} + \frac{2v}{c^2}\frac{\partial^2}{\partial z'\partial t'} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right] f(z',t') = 0 \quad [W']$

which have be solved by $f(z',t') = h(z' - (c-v)t')$ for any scalar function $h(s)$.

This makes sense - consider for example a pulse-like solution of [W] propagating to the right at speed c (in frame K)



in frame K' (moving to the right at speed v) the same pulse will move to the right at speed c−v

if frame K' is moving a speed c, the pulse will be stationary.

Perhaps less obvious is what happens to oscillatory solutions of [W], for example $f(z,t) = \cos[k(z-ct)]$

The corresponding solution in K' is $\cos[k(z-(c-v)t)]$

and if v is chosen to be c this becomes $\cos[kz]$ which does not oscillate in time.

* CONSIDER A SOURCE OF E/M WAVES MOVING WITH

EINSTEIN'S RELATIVITY

Light waves, which are solutions of a wave equation, do not behave as Galilean relativity suggests. We are forced to abandon Galilean relativity and replace it with Einstein's relativity, which can be obtained from two postulates

I. The laws of physics take the same form in every inertial frame

II. The speed of light in vacuum is the same in every inertial frame

We can easily show that these postulates destroy the notion of absolute time and suggest that two events which are simultaneous to one observer are not simultaneous to another. [SHOW TRAIN ANIMATIONS].

We need to find replacements for the Galilean transformations $[\vec{r}' = \vec{r} - \vec{v}t,\ t' = t]$ that are consistent with the two postulates made above.

The most general transformation possible would be

$$x' = x'(x,y,z,t) \qquad y' = y'(x,y,z,t) \qquad z' = z'(x,y,z,t) \qquad t' = t'(x,y,z,t)$$

but we'd like space to be homogeneous and isotropic, so e.g. the infinitesimal displacement in x',

$$dx' = \frac{\partial x'}{\partial x}dx + \frac{\partial x'}{\partial y}dy + \frac{\partial x'}{\partial z}dz + \frac{\partial x'}{\partial t}dt,$$

shouldn't depend on position or time – this implies that the partial derivatives are constants and the transformations must be linear in (x,y,z,t)

Let's return to the frames K, K' we considered previously, where K' moves at constant velocity $v\hat{z}$.

We can focus on (z,t) alone, as rotational invariance is such that $x'=x$, $y'=y$ is the only possible transformation. The most general linear transformation then is

$$z' = Az + Bt \qquad \& \qquad t' = Cz + Dt$$

The origin of K' is at position $z = vt$ by the construction of the frames and thus

$$0 = Avt + Bt \quad \Rightarrow \quad B = -Av$$

so $z' = A(z - vt)$ & $t' = Cz + Dt$

Now suppose a point source of light emits a spherical wavefront at $t=0$ from the origin of K (which at this time is also the origin of K').

This wavefront reaches a point (x,y,z) at a time t satisfying

$$x^2+y^2+z^2 = c^2t^2 \qquad [L]$$

Now postulate II states that in frame K' the speed of the wavefront is also c, so it reaches at point (x',y',z') at a time t' satisfying

$$x'^2+y'^2+z'^2 = c^2t'^2 \qquad [L']$$

Inserting our trial transformations into [L'] we obtain

$$x^2+y^2+A^2(z-vt)^2 = c^2(Cz+Dt)^2$$

$$x^2+y^2+z^2(A^2-c^2C^2) + 2zt(-vA^2 - c^2CD) + t^2(A^2v^2-c^2D^2) = 0$$

but [L] states that $x^2+y^2 = c^2t^2 - z^2$ and thus

$$z^2(A^2-c^2C^2-1) - 2zt(vA^2-c^2CD) + t^2(A^2v^2-c^2D^2+c^2) = 0$$

and this equation must hold for any choices of (z,t)

$$\Rightarrow \quad A^2-c^2C^2=1 \;, \; vA^2=-c^2CD \;, \; A^2v^2-c^2D^2=-c^2$$

eliminating C : $C = -\frac{v}{c^2}\frac{A^2}{D} \Rightarrow A^2 - c^2\frac{v^2}{c^4}\frac{A^4}{D^2} = 1 \Rightarrow -\frac{v^2}{c^2}A^4 + A^2D^2 - D^2 = 0$

$\frac{v^2}{c^2}A^4 - A^2D^2 + D^2 = 0$ / $A^2 = \frac{c^2}{v^2}(D^2-1) \Rightarrow \frac{v^2}{c^2}\cdot\left(\frac{c^2}{v^2}\right)^2(D^2-1)^2 - \frac{c^2}{v^2}D^2(D^2-1) + D^2 = 0$

$$\frac{c^2}{v^2}\left[\cancel{D^4}-2D^2+1-\cancel{D^4}+D^2\right] = -D^2$$

$$\frac{c^2}{v^2}\left[1-D^2\right] = -D^2 \Rightarrow -D^2\left(1-\frac{c^2}{v^2}\right) = \frac{c^2}{v^2} \Rightarrow D^2 = \frac{c^2/v^2}{c^2/v^2-1} = \frac{1}{1-v^2/c^2}$$

then $A^2 = \frac{c^2}{v^2}(D^2-1) = \frac{c^2}{v^2}\cdot\frac{v^2}{c^2}\frac{1}{1-v^2/c^2} = \frac{1}{1-v^2/c^2}$

$$A^2 = D^2 = \frac{1}{1 - v^2/c^2}$$

we need the positive square root so that in the limit $v \ll c$ we recover the Galilean result $z' = z - vt$

$$A = D = \frac{1}{\sqrt{1 - v^2/c^2}} \qquad \& \quad C = -\frac{v}{c^2}\frac{1}{\sqrt{1-v^2/c^2}}$$

and defining some convenient symbols, $\beta \equiv v/c$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, we have

$$\boxed{z' = \gamma(z - \beta ct) \quad \& \quad ct' = \gamma(ct - \beta z)}$$ the LORENTZ transformations

the inverse transformations can be obtained by taking $v \to -v$

$$z = \gamma(z' + \beta ct') \qquad ct = \gamma(ct' + \beta z')$$

check by substitution

$$z = \gamma\left(\gamma(z - \beta ct) + \beta\gamma(ct - \beta z)\right)$$
$$= (\gamma^2 - \beta^2\gamma^2)z + (-\beta\gamma^2 + \beta\gamma^2)ct$$
$$= z \checkmark$$

$$ct = \gamma\left(\gamma(ct - \beta z) + \beta\gamma(z - \beta ct)\right)$$
$$= (\gamma^2 - \beta^2\gamma^2)ct + (-\beta\gamma^2 + \beta\gamma^2)z$$
$$= ct \checkmark$$

Clearly postulate II does not allow the Galilean addition of velocities formula to remain true ($u' = u - v$). The Lorentz transformations imply a different relation

$$u' = \frac{dz'}{dt'} = \frac{dz'}{dt}\frac{dt}{dt'} = \gamma\left(\frac{dz}{dt} - v\right)\gamma\left(1 + \frac{v}{c}\frac{dz'}{dt'}\right)$$

$$u' = \gamma^2(u - v)\left(1 + \frac{vu'}{c^2}\right) \Rightarrow u'\left(1 - \gamma^2\frac{v}{c^2}(u - v)\right) = \gamma^2(u - v)$$

$$u'\left(1/\gamma^2 - \frac{vu}{c^2} + v^2/c^2\right) = u - v$$

$$u'\left(1 - v^2/c^2 - \frac{vu}{c^2} + v^2/c^2\right) = u - v \Rightarrow \boxed{u' = \frac{u - v}{1 - \frac{uv}{c^2}}} \xrightarrow{u, v \ll c} u - v \quad \checkmark \text{Galilean}$$

$$\xrightarrow{u = c} \frac{c - v}{1 - v/c} = c \quad \checkmark \text{postulate II}$$

It's worth going back to examine the wave equation under a Lorentz transformation:

$$\frac{\partial}{\partial z} = \gamma \frac{\partial}{\partial z'} - \frac{\gamma\beta}{c}\frac{\partial}{\partial t'} \qquad \& \qquad \frac{\partial}{\partial t} = -\gamma\beta c \frac{\partial}{\partial z'} + \gamma \frac{\partial}{\partial t'}$$

$$\Rightarrow \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \left(\gamma^2 - \gamma^2\beta^2\right)\frac{\partial^2}{\partial z'^2} + \left(-\frac{2\gamma^2\beta}{c} + \frac{2\gamma^2\beta}{c}\right)\frac{\partial^2}{\partial z'\partial t} + \left(\gamma^2\beta^2/c^2 - \gamma^2/c^2\right)\frac{\partial^2}{\partial t'^2}$$

$$= \frac{\partial^2}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}$$

so the solutions $f(z,t) = h(z-ct)$ simply become solutions $f(z',t') = h(z'-ct')$
i.e. waves travelling at the same speed c in <u>any frame</u>.

the invariant interval

consider two "events" whose co-ordinates in K are (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) then we may define an invariant quantity,

$$(\Delta s)^2 \equiv c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \qquad \text{where} \quad \Delta x \equiv x_2 - x_1 \ \text{etc...}$$

We can show that $(\Delta s)^2$ takes the same value in any frame. For example, considering the frame K':

$$(\Delta s')^2 \equiv c^2(\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2$$

$$= \gamma^2(c\Delta t - \beta \Delta z)^2 - (\Delta x)^2 - (\Delta y)^2 - \gamma^2(\Delta z - \beta c\Delta t)^2$$

$$= \gamma^2(1-\beta^2)c^2(\Delta t)^2 + (2\gamma^2\beta c - 2\gamma^2\beta c)\Delta t \Delta z - \gamma^2(1-\beta^2)(\Delta z)^2$$

$$= c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = (\Delta s)^2$$

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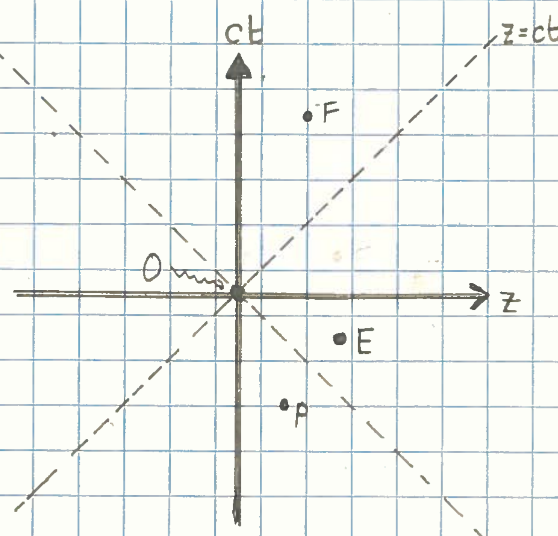
three classes of event separation are

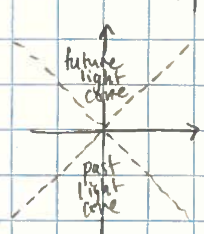
$(\Delta s)^2 < 0$ "space-like"

$(\Delta s)^2 = 0$ "null" or "light-like"

$(\Delta s)^2 > 0$ "time-like"



Consider the space-time diagram



the invariant interval is an important tool in determining the CAUSAL connection between two events, recalling that the time interval depends upon the frame of reference of the observer.

e.g. consider the events O & P – in frame K, $\Delta t<0$ since $\Delta t = t_P - t_O$ & $t_P < t_O$ and we'd say that P is in the PAST of O.

The corresponding invariant interval $(\Delta s)^2 > 0$ since $|\Delta z| < c|\Delta t|$
$\rightarrow$ a "time-like" separation

Let's try however to boost to a frame K' in which $\Delta t' > 0$ so that P is in the FUTURE of O:

$$c\Delta t' = \gamma(c\Delta t - \beta\Delta z) = \gamma c\Delta t\left(1 - \frac{\beta}{c}\frac{\Delta z}{\Delta t}\right)$$

Since $\Delta t<0$, we need $\frac{\beta}{c}\left|\frac{\Delta z}{\Delta t}\right| > 1$ in order to achieve this

i.e. $\beta > c\Big/\left|\frac{\Delta z}{\Delta t}\right|$

but we know that $|\Delta z| < c|\Delta t|$ $\Rightarrow$ $\frac{|\Delta z|}{|\Delta t|} < c$ & thus we'd need $\beta > 1$, which is not possible

The same logic shows that event F is in the future of event O in any frame of reference.

CAUSALITY is meaningful for events with a time-like separation.

Conversely, consider the events E & O. Since E occurs at an earlier time than O we might be tempted to conclude that E is in the past of O. We can show that this is a frame-dependent statement:
$\Delta t < 0$

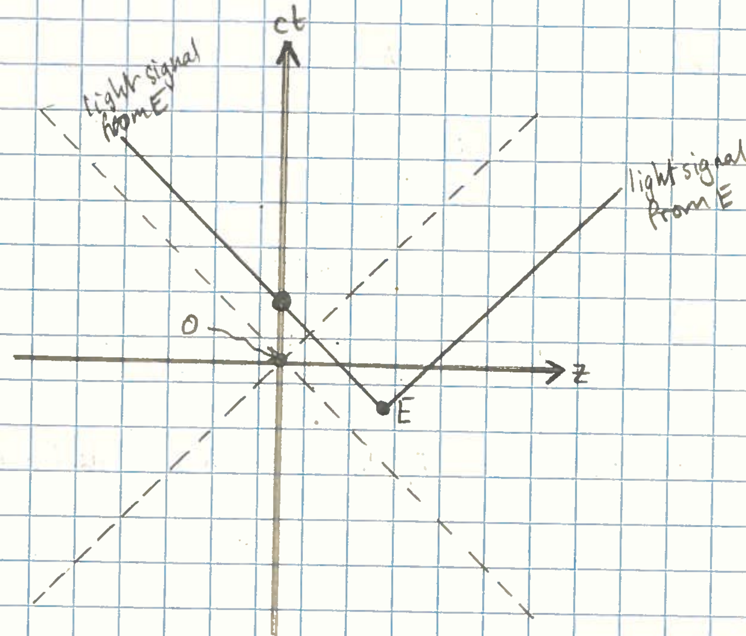
Firstly realize that the separation is "space-like" $(\Delta s)^2 < 0$ since $|\Delta z| > c|\Delta t|$

Now let's try to boost to a frame where $\Delta t' > 0$

$$c\Delta t' = \gamma c\Delta t\left(1 - \beta/c\,\frac{\Delta z}{\Delta t}\right) \Rightarrow \text{want } \frac{\beta}{c}\frac{\Delta z}{\Delta t} > 1 \text{ or } \beta > c\Big/\left|\frac{\Delta z}{\Delta t}\right|$$

but since $|\Delta z| > c|\Delta t|$, $\left|\frac{\Delta z}{\Delta t}\right| > c$ & $\beta < 1$ & there are frames in which E occurs later than O.

Isn't this a problem for cause & effect? We can see that it isn't by trying to influence O from E – the fastest signal we can send is via a light wave travelling at speed c



and we see that the earliest time that a light signal from E can influence something located at $z=0$ is at a time LATER than O.

Equally a light signal from O can't influence E, and E is neither in the past of O, nor in its future – it is "elsewhere".

the proper time

another way of expressing the invariant interval is in terms of the "proper time" – consider an object moving with velocity $\vec{u}(t) = \frac{d\vec{r}}{dt}$ in frame K, then the interval for two infinitesimally seperated points on its trajectory is

$$(ds)^2 = (c\,dt)^2 - (d\vec{r})^2 = (c\,dt)^2\left[1 - \frac{1}{c^2}\left(\frac{d\vec{r}}{dt}\right)^2\right] = (c\,dt)^2\left[1 - u^2(t)/c^2\right]$$

and we define the PROPER TIME $\quad d\tau \equiv \sqrt{\frac{(ds)^2}{c^2}} = dt\sqrt{1 - u^2(t)/c^2} = \frac{dt}{\gamma(u)}$

it follows that if we stay in a frame in which the object is instantaneously at rest, $u=0$, $\gamma=1$ and $d\tau = dt$, so we interpret the proper time as the time measured by the objects "own" clock.

thus far we've assumed that the boost from frame K to frame K' is along the z-axis - more generally let's allow it to be in an arbitrary direction with velocity $\vec{v}$.

Defining $\vec{\beta} = \vec{v}/c$ the Lorentz transformation is

$$\vec{r}_\perp' = \vec{r}_\perp$$

$$\vec{r}_\parallel' = \gamma\left[\vec{r}_\parallel - \vec{\beta} c t\right]$$

$$ct' = \gamma\left[ct - \vec{\beta}\cdot\vec{r}_\parallel\right]$$

where $\vec{r} = \vec{r}_\perp + \vec{r}_\parallel$ with $\vec{r}_\parallel$ the component parallel to $\vec{\beta}$

FOUR VECTORS

The Lorentz transformations suggest that we need a uniform treatment of space and time together, and it will prove useful to build an object called the "space-time" four vector

x^μ ["contravariant", "raised index"]

whose components are $x^\mu = [x^0, x^1, x^2, x^3] \equiv [ct, x, y, z] = [ct, \vec{r}]$

A useful, closely related object, the "covariant" or "lowered index" space-time four vector is

$x_\mu = [x_0, x_1, x_2, x_3] = [ct, -x, -y, -z] = [ct, -\vec{r}]$

Taken together we can define a SCALAR PRODUCT, e.g. for the difference between two four-vectors as

$$\Delta x_\mu \Delta x^\mu = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

which we identify as the invariant interval $(\Delta s)^2$.

The relationship between x_μ and x^μ is via a tensor called the METRIC TENSOR, $g_{\mu\nu}$

$x_\mu = g_{\mu\nu} x^\nu$

the metric tensor can be represented by the matrix $g_{\mu\nu} \rightsquigarrow \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 \end{bmatrix}$

The inverse transformation, $x^\mu = g^{\mu\nu} x_\nu$, features $g^{\mu\nu}$, whose matrix representation we can find:

$x^\mu = g^{\mu\nu} x_\nu = g^{\mu\nu} g_{\nu\alpha} x^\alpha$ so clearly we need $g^{\mu\nu}$'s matrix to be the inverse of $g_{\mu\nu}$

but $g_{\mu\nu}$ is its own inverse; $g^{\mu\nu} \rightsquigarrow \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 \end{bmatrix}$

$g^{\mu\nu} g_{\nu\alpha} = \delta^\mu{}_\alpha$

& notice that if we consider $g^{\mu\nu}$ to always raise the ν index to μ of whatever it acts on $g^{\mu\nu} g_{\nu\alpha} = g^\mu{}_\alpha = \delta^\mu{}_\alpha \rightsquigarrow \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \vdots & & & \end{bmatrix}$

We can express the Lorentz transforms in terms of the action of a tensor on x^μ.

Choosing the boost along z for simplicity we recall that $z' = \gamma[z - \beta ct]$ & $ct' = \gamma[ct - \beta z]$ & $z = \gamma[z' + \beta ct']$, $ct = \gamma[ct' + \beta z']$

$$x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} x^\nu = L^\mu{}_\nu x^\nu$$

with $x^\nu = [ct, x, y, z]$

$$\Rightarrow L^\mu{}_\nu \sim \begin{bmatrix} \gamma & \cdot & \cdot & -\beta\gamma \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ -\beta\gamma & \cdot & \cdot & \gamma \end{bmatrix}$$

N.B. symmetric matrix

the inverse transform $x^\mu = \frac{\partial x^\mu}{\partial x'^\nu} x'^\nu = [L^{-1}]^\mu{}_\nu x'^\nu$

$$[L^{-1}]^\mu{}_\nu L^\nu{}_\alpha = \delta^\mu{}_\alpha$$

where $[L^{-1}]^\mu{}_\nu \sim \begin{bmatrix} \gamma & \cdot & \cdot & \beta\gamma \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \beta\gamma & \cdot & \cdot & \gamma \end{bmatrix}$

The transformations for the "lowered index" objects can be obtained using the invariance of $x_\mu x^\mu$

$$x'_\mu x'^\mu = x_\mu x^\mu$$

$$x'_\mu x'^\mu = x'_\mu \frac{\partial x'^\mu}{\partial x^\nu} x^\nu \qquad \Rightarrow x_\nu = x'_\mu \frac{\partial x'^\mu}{\partial x^\nu} = x'_\mu L^\mu{}_\nu$$

$$\Rightarrow x_\nu [L^{-1}]^\nu{}_\alpha = x'_\mu L^\mu{}_\nu [L^{-1}]^\nu{}_\alpha = x'_\mu \delta^\mu{}_\alpha = x'_\alpha$$

So $x'_\mu = x_\nu [L^{-1}]^\nu{}_\mu$ & similarly the inverse transformation $x_\mu = x'_\nu L^\nu{}_\mu$

Any object a^μ that transforms this way under a Lorentz transformation is called a four-vector

$$\boxed{\begin{array}{ll} a'^\mu = L^\mu{}_\nu a^\nu & a^\mu = [L^{-1}]^\mu{}_\nu a'^\nu \\ a'_\mu = a_\nu [L^{-1}]^\nu{}_\mu & a_\mu = a'_\nu L^\nu{}_\mu \end{array}}$$

these transformation properties ensure that the scalar product of any two four-vectors is invariant:

$$a_\mu b^\mu = a'_\mu b'^\mu$$

space-time derivatives are an important feature of our dynamical equations

e.g. differentiation w.r.t the components of the "raised index" four-vector x^μ

$$\frac{\partial}{\partial x^\mu} = \left[\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] = \left[\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right]$$

How does this transform under a Lorentz transformation?

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = [L^{-1}]^\nu{}_\mu \frac{\partial}{\partial x^\nu} = \frac{\partial}{\partial x^\nu}[L^{-1}]^\nu{}_\mu$$

and we see that $\frac{\partial}{\partial x^\mu}$ transforms like a "lowered index" four-vector

$$\Rightarrow \quad \partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left[\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right]$$

The corresponding "raised index" four-vector is $\partial^\mu = g^{\mu\nu}\partial_\nu = \left[\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right]$.

It follows that we can write invariants $\partial_\mu a^\mu = \frac{1}{c}\frac{\partial a^0}{\partial t} + \vec{\nabla}\cdot\vec{a}$

and $\quad \partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$

[note that the scalar wave eqn without sources becomes $\partial_\mu\partial^\mu f = 0$ & its invariance under boosts is now obvious]

Looking ahead we'll also propose that there are 2^{nd} rank LORENTZ TENSORs which carry two indices & which transform as

$$A'^{\mu\nu} = L^\mu{}_\alpha L^\nu{}_\beta A^{\alpha\beta}$$

$$A'_{\mu\nu} = A_{\alpha\beta}[L^{-1}]^\alpha{}_\mu [L^{-1}]^\beta{}_\nu$$

we've already seen one example of such a tensor, $g^{\mu\nu}$. $\quad g'^{\mu\nu} = L^\mu{}_\alpha L^\nu{}_\beta g^{\alpha\beta}$

$$\left.\begin{aligned} a'^\mu &= L^\mu{}_\nu a^\nu = L^\mu{}_\nu g^{\nu\beta} a_\beta \\ a'^\mu &= g^{\mu\rho} a'_\rho = g^{\mu\rho} a_\beta [L^{-1}]^\beta{}_\rho \end{aligned}\right\} \; L^\mu{}_\nu g^{\nu\beta} = [L^{-1}]^\beta{}_\rho g^{\mu\rho}$$

$$\Rightarrow \; L^\alpha{}_\beta L^\mu{}_\nu g^{\nu\beta} = L^\alpha{}_\beta [L^{-1}]^\beta{}_\rho g^{\mu\rho}$$

MECHANICAL QUANTITIES

We might try now to construct a four-vector which will be a covariant version of the velocity, a so-called FOUR-VELOCITY.

Let's define $U^\mu = \dfrac{dx^\mu}{d\tau}$ where the time derivative is with respect to the proper time, τ.

then $U^\mu = \gamma(u)\dfrac{d}{dt}[ct, \vec{r}] = \gamma(u)[c, \vec{u}]$ so $U^0 = c\gamma(u)$, $\vec{U} = \vec{u}\,\gamma(u)$

and $U_\mu U^\mu = \gamma^2(c^2 - \vec{u}\cdot\vec{u}) = \dfrac{c^2 - \vec{u}\cdot\vec{u}}{1 - \frac{\vec{u}\cdot\vec{u}}{c^2}} = c^2$ $\quad U_\mu U^\mu = c^2 \Rightarrow U^\mu$ is a "time-like" four-vector.

Consider a boost from frame K to frame K' along the z-axis we have

$$\gamma(u')\begin{bmatrix} c \\ u'_x \\ u'_y \\ u'_z \end{bmatrix} = \begin{bmatrix} \gamma_v & \cdot & \cdot & -\beta_v\gamma_v \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ -\beta_v\gamma_v & \cdot & \cdot & \gamma_v \end{bmatrix}\begin{bmatrix} c \\ u_x \\ u_y \\ u_z \end{bmatrix}\gamma(u)$$

$$\Rightarrow \gamma(u')c = \gamma(u)\gamma(v)\left(c - \frac{v}{c}u_z\right) \quad \Rightarrow \quad \frac{\gamma(u)}{\gamma(u')} = \frac{1}{\gamma(v)}\frac{1}{1 - \frac{vu_z}{c^2}} \quad [G]$$

$$\gamma(u')u'_x = \gamma(u)u_x$$

$$\gamma(u')u'_y = \gamma(u)u_y$$

$$\gamma(u')u'_z = \gamma(u)\gamma(v)\left(-\frac{v}{c}c + u_z\right)$$

and using eqn [G] to eliminate $\gamma(u)$ and $\gamma(u')$ from the last three equations we have

$$u'_x = \frac{1}{\gamma(v)}\frac{u_x}{1 - vu_z/c^2} \qquad u'_y = \frac{1}{\gamma(v)}\frac{u_y}{1 - vu_z/c^2}$$

$$u'_z = \frac{u_z - v}{1 - vu_z/c^2}$$ which agrees with our previous result in the case of motion only in the z-direction.

four-momentum

It's more common when dealing with the mechanics of relativistic particles to work with the four-momentum rather than the four-velocity.

The four-momentum may be defined as

$$p^\mu = m U^\mu$$ where m is a Lorentz scalar whose identity we will determine

the three-vector components of this four-vector are

$$p^i = \gamma(u) m u^i = m u^i \frac{1}{\sqrt{1-u^2/c^2}} = m u^i \left(1 + \frac{1}{2}\frac{u^2}{c^2} + \ldots\right)$$

and we see that in the limit $u \ll c$, this quantity is indeed the usual momentum provided m is identified as the mass of the particle.

the "temporal" component is

$$p^0 = \gamma(u) mc = \frac{1}{c}\left[\gamma(u) mc^2\right] = \frac{1}{c}\left[\underbrace{mc^2}_{\text{the "rest mass energy"}} + \underbrace{\frac{1}{2}mu^2}_{\text{non-rel kinetic energy}} + \frac{3}{8}m\frac{u^4}{c^2} + \ldots\right]$$

the object in square brackets $\gamma(u)mc^2$ can sensibly be termed the energy of the particle, E

Notice the role played by m in the invariant norm of p^μ:

$$p_\mu p^\mu = m^2 U_\mu U^\mu = m^2 c^2$$

Which gives us a dispersion relation between $|\vec{p}|$ and E:

$$(E/c)^2 - \vec{p}\cdot\vec{p} = m^2c^2 \longrightarrow \underline{E^2 - p^2c^2 = m^2c^4}$$

although it is not obvious from the derivation presented, zero mass particles can also be described by a four-momentum with non-zero entries – in that case

$$E = |\vec{p}|c \qquad \& \quad p^\mu = [p, \vec{p}]$$

We might guess that the relativistic generalization of the equation of motion for a charged particle in $\vec{E}, \vec{B}$ fields is

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}\left[\gamma(u) m \vec{u}\right] = q\left(\vec{E} + \vec{u} \times \vec{B}\right)$$

we'll show later that this is correct.

note also the rate of change of particle energy, $\frac{dE}{dt}$

$$E^2 - p^2c^2 = m^2c^4 \quad \Rightarrow \quad 2E\frac{dE}{dt} = c^2\, 2\vec{p}\cdot\frac{d\vec{p}}{dt} \quad \Rightarrow \quad \frac{dE}{dt} = c^2\, \frac{\vec{p}}{E}\cdot\frac{d\vec{p}}{dt} = c^2\, \frac{\gamma m\, \vec{u}}{\gamma m^2}\cdot\frac{d\vec{p}}{dt} = \vec{u}\cdot\frac{d\vec{p}}{dt}$$

$$\frac{dE}{dt} = \vec{u}\cdot\left[q\left(\vec{E} + \vec{u}\times\vec{B}\right)\right] = q\,\vec{u}\cdot\vec{E}$$

as we might expect.

elativity and electromagnetism

First let's consider the sources for e/m fields, charge & current, satisfying a continuity equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 .$$

The form of this equation immediately suggests a four-vector $J^\mu = [c\rho, \vec{J}]$

Since then $\partial_\mu J^\mu = 0$ and we have that charge is conserved for all observers.

Observe that, as we might expect, what is charge to one observer is current to another (and vice versa)

$$c\rho' = \gamma(c\rho - \vec{\beta}\cdot\vec{J}_{\parallel}) \quad ; \quad \vec{J}_\perp{}' = \vec{J}_\perp \quad ; \quad \vec{J}_{\parallel}{}' = \gamma(\vec{J}_{\parallel} - \vec{\beta}c\rho)$$

$$\& \ c\rho = \gamma(c\rho' + \vec{\beta}\cdot\vec{J}_{\parallel}{}') \quad ; \quad \vec{J}_\perp{}' = \vec{J}_\perp \quad ; \quad \vec{J}_{\parallel} = \gamma(\vec{J}_{\parallel}{}' + \vec{\beta}c\rho')$$

for example suppose a charge distribution is a rest in frame K', then $\vec{J}_\perp{}' = 0$, $\vec{J}_{\parallel}{}' = 0$

and $\rho = \rho'\cdot\gamma$, $\vec{J}_{\parallel} = \gamma\vec{\beta}c\rho' = \vec{v}\rho$

so in frame K the density of charge is increased by a factor γ and there is a current $\vec{J} = \rho\vec{v}$.

Note that the total charge in an infinitesimal volume is invariant:

$$dq = \rho\, dx\, dy\, dz$$

$$dq' = \rho' dx' dy' dz' \qquad \text{but } \rho' = \rho/\gamma ,\ dx = dx',\ dy = dy',\ dz' = \gamma dz$$

$$= \rho dx dy dz \, \gamma/\gamma = dq \ \checkmark$$

the scalar and vector potentials in LORENZ GAUGE form a four-vector -

recalling the Lorenz gauge condition $\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0$

we may define $A^\mu = [\varphi/c, \vec{A}]$ satisfying $\underline{\partial_\mu A^\mu = 0}$

The potentials satisfy $\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\varphi = -\rho/\varepsilon_0$

& $\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{A} = -\mu_0\vec{J}$

which can be written in covariant form as $\partial_\mu\partial^\mu A^\nu = \mu_0 J^\nu$

and since $\partial_\mu\partial^\mu$ is a Lorentz invariant, if J^ν is a four-vector, so is A^ν.

The electric and magnetic fields are three-vectors, $\vec{E}, \vec{B}$ — do they form part of a pair of four vectors, $E^\mu = (?, \vec{E})$ & $B^\mu(?, \vec{B})$?

The elegant way to explore this is to use covariant notation, but to teach ourselves a lesson, let's do it the inelegant way!

We'll try to find the transformation properties of $\vec{E}$ and $\vec{B}$ under Lorentz transformations: the answer will turn out to be

$$\vec{E}'_\parallel = \vec{E}_\parallel \qquad \vec{E}'_\perp = \gamma\left(\vec{E}_\perp + c\,\vec{\beta}\times\vec{B}_\perp\right)$$
$$\vec{B}'_\parallel = \vec{B}_\parallel \qquad \vec{B}'_\perp = \gamma\left(\vec{B}_\perp - \frac{1}{c}\vec{\beta}\times\vec{E}_\perp\right) \qquad [L]$$

and we see that electric fields in one frame are magnetic fields in another. These are not the transformations of the 3-vectors parts of a four-vector — it will turn out that a 2^{nd} rank Lorentz tensor actually houses both $\vec{E}$ & $\vec{B}$.

First let's derive equations [L] starting from the transformation properties of a known four-vector, the four-potential $A^\mu = [\varphi/c, \vec{A}]$,

and the fact that $\vec{B} = \vec{\nabla}\times\vec{A}$, $\vec{E} = -\vec{\nabla}\varphi - \frac{\partial\vec{A}}{\partial t}$

the transformation properties of ∂_μ indicate that

$$\vec{\nabla}_\perp' = \vec{\nabla}_\perp \quad / \quad \vec{\nabla}_\parallel' = \gamma\left(\vec{\nabla}_\parallel + \frac{1}{c}\vec{\beta}\frac{\partial}{\partial t}\right) \quad / \quad \frac{\partial}{\partial t'} = \gamma\left(c\vec{\beta}\cdot\vec{\nabla}_\parallel + \frac{\partial}{\partial t}\right)$$

and for A^μ

$$\vec{A}_\perp' = \vec{A}_\perp \quad / \quad \vec{A}_\parallel' = \gamma(\vec{A}_\parallel - \vec{\beta}\varphi/c) \quad / \quad \varphi' = \gamma(\varphi - c\vec{\beta}\cdot\vec{A}_\parallel)$$

so $\vec{B}' = \vec{\nabla}'\times\vec{A}' = (\vec{\nabla}_\perp' + \vec{\nabla}_\parallel')\times(\vec{A}_\perp' + \vec{A}_\parallel') = \vec{\nabla}_\perp'\times\vec{A}_\perp' + \vec{\nabla}_\perp'\times\vec{A}_\parallel' + \vec{\nabla}_\parallel'\times\vec{A}_\perp'$

$$\vec{B}_\parallel' = \vec{\nabla}_\perp'\times\vec{A}_\perp' \quad \Rightarrow \quad \boxed{\vec{B}_\parallel' = \vec{B}_\parallel}$$

$$\vec{B}_\perp' = \vec{\nabla}_\perp'\times\vec{A}_\parallel' + \vec{\nabla}_\parallel'\times\vec{A}_\perp'$$

$$= \gamma\vec{\nabla}_\perp\times(\vec{A}_\parallel - \vec{\beta}\varphi/c) + \gamma\left(\vec{\nabla}_\parallel + \frac{1}{c}\vec{\beta}\frac{\partial}{\partial t}\right)\times\vec{A}_\perp$$

$$= \gamma\left[\vec{\nabla}_\perp\times\vec{A}_\parallel + \vec{\nabla}_\parallel\times\vec{A}_\perp + \frac{1}{c}\vec{\beta}\times\vec{\nabla}_\perp\varphi + \frac{1}{c}\vec{\beta}\times\frac{\partial\vec{A}_\perp}{\partial t}\right]$$

$$\boxed{\vec{B}_\perp' = \gamma\left[\vec{B}_\perp - \frac{1}{c}\vec{\beta}\times\vec{E}_\perp\right]}$$

$$\vec{E}_\parallel' = -\vec{\nabla}_\parallel'\varphi' - \frac{\partial}{\partial t'}\vec{A}_\parallel' = -\gamma^2\left[\left(\vec{\nabla}_\parallel + \frac{1}{c}\vec{\beta}\frac{\partial}{\partial t}\right)(\varphi - c\vec{\beta}\cdot\vec{A}_\parallel) + \left(c\vec{\beta}\cdot\vec{\nabla}_\parallel + \frac{\partial}{\partial t}\right)\left(\vec{A}_\parallel - \frac{1}{c}\vec{\beta}\varphi\right)\right]$$

$$= -\gamma^2\left\{\vec{\nabla}_\parallel\varphi - c\vec{\nabla}_\parallel(\vec{\beta}\cdot\vec{A}_\parallel) + \cancel{\frac{1}{c}\vec{\beta}\frac{\partial\varphi}{\partial t}} - \vec{\beta}\,\vec{\beta}\cdot\frac{\partial\vec{A}_\parallel}{\partial t} + c(\vec{\beta}\cdot\vec{\nabla}_\parallel)\vec{A}_\parallel - \vec{\beta}(\vec{\beta}\cdot\vec{\nabla}_\parallel)\varphi + \frac{\partial\vec{A}_\parallel}{\partial t} - \cancel{\frac{1}{c}\vec{\beta}\frac{\partial\varphi}{\partial t}}\right]$$

$$\left\{\text{N.B. } \vec{\beta}\,\vec{\beta}\cdot\vec{X}_\parallel = \beta^2\vec{X}_\parallel\right\}$$

$$= -\gamma^2\left\{(1-\beta^2)\vec{\nabla}_\parallel\varphi + (1-\beta^2)\frac{\partial\vec{A}_\parallel}{\partial t} + c(\vec{\beta}\cdot\vec{\nabla}_\parallel)\vec{A}_\parallel - c\left(\vec{\beta}\cdot\vec{\nabla}_\parallel\vec{A}_\parallel + \vec{\beta}\times(\vec{\nabla}_\parallel\times\vec{A}_\parallel)\right)\right\}$$

$$= -\vec{\nabla}_\parallel\varphi - \frac{\partial\vec{A}_\parallel}{\partial t} \quad \Rightarrow \quad \boxed{\vec{E}_\parallel' = \vec{E}_\parallel}$$

$$\vec{E}_\perp' = -\vec{\nabla}_\perp'\varphi' - \frac{\partial\vec{A}_\perp'}{\partial t'} = -\gamma\left[+\vec{\nabla}_\perp(\varphi - c\vec{\beta}\cdot\vec{A}_\parallel) + \left(c\vec{\beta}\cdot\vec{\nabla}_\parallel + \frac{\partial}{\partial t}\right)\vec{A}_\perp\right]$$

$$= \gamma\left[-\vec{\nabla}_\perp\varphi - \frac{\partial\vec{A}_\perp}{\partial t} + c\vec{\nabla}_\perp(\vec{\beta}\cdot\vec{A}_\parallel) - c(\vec{\beta}\cdot\vec{\nabla}_\parallel)\vec{A}_\perp\right]$$

$$= \gamma\left[\vec{E}_\perp + c\underbrace{\vec{\beta}\cdot\vec{\nabla}_\perp}_{=0}\vec{A}_\parallel + c\vec{\beta}\times(\vec{\nabla}_\perp\times\vec{A}_\parallel) - c(\vec{\beta}\cdot\vec{\nabla}_\parallel)\vec{A}_\perp\right] \qquad \left\{\vec{\nabla}_\parallel(\underbrace{\vec{\beta}\cdot\vec{A}_\perp}_{=0}) = (\vec{\beta}\cdot\vec{\nabla}_\parallel)\vec{A}_\perp + \vec{\beta}\times(\vec{\nabla}_\parallel\times\vec{A}_\perp)\right.$$

$$= \gamma\left[\vec{E}_\perp + c\vec{\beta}\times(\vec{\nabla}_\perp\times\vec{A}_\parallel + \vec{\nabla}_\parallel\times\vec{A}_\perp)\right]$$

$$\boxed{\vec{E}_\perp' = \gamma\left[\vec{E}_\perp + c\vec{\beta}\times\vec{B}_\perp\right]}$$

point charge in uniform motion

sider a point charge which is at rest in frame K' - in that frame there will be magnetic field, $\vec{B}'=\vec{0}$, and the electric field is given by Coulomb's law

$$\vec{E}'(\vec{r}',t') = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}'}{r'^3} = \frac{q}{4\pi\epsilon_0}\left[\frac{\vec{r}_{\parallel}'}{(\vec{r}_{\parallel}'\cdot\vec{r}_{\parallel}'+\vec{r}_{\perp}'\cdot\vec{r}_{\perp}')^{3/2}} + \frac{\vec{r}_{\perp}'}{(\vec{r}_{\parallel}'\cdot\vec{r}_{\parallel}'+\vec{r}_{\perp}'\cdot\vec{r}_{\perp}')^{3/2}}\right] = \vec{E}_{\parallel}' + \vec{E}_{\perp}'$$

nce frame K' is moving at a constant velocity $\vec{v}$ with respect to frame K, in frame the point charge is moving at constant velocity $\vec{v}$.

The fields produced by the point charge in frame K are

$$\vec{E} = \vec{E}_{\parallel} + \vec{E}_{\perp} = \vec{E}_{\parallel}' + \gamma\vec{E}_{\perp}'$$

and $\vec{B} = \frac{1}{c}\vec{\beta}\times\vec{E}_{\perp}'$

since q is invariant, $\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}_{\parallel}' + \gamma\vec{r}_{\perp}'}{(\vec{r}_{\parallel}'\cdot\vec{r}_{\parallel}'+\vec{r}_{\perp}'\cdot\vec{r}_{\perp}')^{3/2}}$

or in terms of the frame K variables

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\gamma(\vec{r}_{\parallel}-\vec{v}t) + \gamma\vec{r}_{\perp}}{(\gamma^2(\vec{r}_{\parallel}-\vec{v}t)^2 + \vec{r}_{\perp}\cdot\vec{r}_{\perp})^{3/2}}$$

which simplifies if we write it in terms of the vector $\vec{R}=\vec{r}-\vec{v}t$ which points from the position of the charge at time t to the observation point

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\gamma\vec{R}}{(\gamma^2\vec{R}_{\parallel}^2 + \vec{R}_{\perp}^2)^{3/2}}$$

and defining an angle by $\hat{v}\cdot\hat{R} = \cos\theta$ we have $\vec{R}_{\parallel} = R\cos\theta$, $|\vec{R}_{\perp}| = R\sin\theta$

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\gamma\vec{R}}{R^3(\gamma^2\cos^2\theta+\sin^2\theta)^{3/2}} = \frac{q}{4\pi\epsilon_0}\frac{\gamma\vec{R}}{R^3(\gamma^2+(1-\gamma^2)\sin^2\theta)^{3/2}} = \frac{q}{4\pi\epsilon_0}\frac{\gamma\vec{R}}{\gamma^3R^3(1-\beta^2\sin^2\theta)^{3/2}}$$

$$\boxed{\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{R}}{R^2}\cdot\frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}}}$$

e.g. forward & backward directions $(\theta\to 0)$ $(\theta\to\pi)$ $\sim 1-\beta^2 \xrightarrow[\beta\to 1]{} 0$

transverse points $\sim \frac{1}{\sqrt{1-\beta^2}} \xrightarrow[\beta\to 1]{} \infty$

Covariant electromagnetism

We've already seen that the wave equations for the Lorenz gauge potentials can be written in form that makes clear their covariance under Lorentz transformations: $\partial_\nu \partial^\nu A^\mu = \mu_0 J^\mu$

where $A^\mu = [\varphi/c, \vec{A}]$ and $J^\mu = [c\rho, \vec{J}]$ are "four-vectors"

Let's now consider the $\vec{E}, \vec{B}$ fields and try to find covariant expressions that express Maxwell's eqns

$\vec{E} = -\vec{\nabla}\varphi - \dfrac{\partial \vec{A}}{\partial t}$ & $\vec{B} = \vec{\nabla} \times \vec{A}$ and recall that $\partial_\mu = \dfrac{\partial}{\partial x^\mu} = \left[\dfrac{1}{c}\dfrac{\partial}{\partial t}, \vec{\nabla}\right]$

so $\partial_0 = \dfrac{1}{c}\dfrac{\partial}{\partial t}$, $\partial_1 = \dfrac{\partial}{\partial x}$, $\partial_2 = \dfrac{\partial}{\partial y}$, $\partial_3 = \dfrac{\partial}{\partial z}$

$\Rightarrow$ $\vec{E}/c = -\vec{\nabla} A^0 - \dfrac{1}{c}\dfrac{\partial}{\partial t}\vec{A}$

$\dfrac{1}{c} E_x = -\dfrac{\partial}{\partial x} A^0 - \dfrac{1}{c}\dfrac{\partial}{\partial t} A_x$ etc...

$A^0 = A_0$ | $A_x = A^1 = -A_1$

so $\dfrac{1}{c} E_x = -\partial_1 A_0 + \partial_0 A_1$

$B_x = \dfrac{\partial}{\partial y} A_z - \dfrac{\partial}{\partial x} A_y = \partial_2(-A_3) - \partial_3(-A_2)$ $\Rightarrow$ $B_x = -\partial_2 A_3 + \partial_3 A_2$ etc...

We can define a rank-two antisymmetric "field-strength" tensor

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

then

$F_{00} = 0$	$F_{01} = \partial_0 A_1 - \partial_1 A_0 = E_x/c$	$F_{02} = E_y/c$	$F_{03} = E_z/c$
$F_{10} = -F_{01} = -E_x/c$	$F_{11} = 0$	$F_{12} = \partial_1 A_2 - \partial_2 A_1 = -B_z$	$F_{13} = \partial_1 A_3 - \partial_3 A_1 = B_y$
$F_{20} = -F_{02} = -E_y/c$	$F_{21} = -F_{12} = B_z$	$F_{22} = 0$	$F_{23} = -B_x$
$F_{30} = -F_{03} = -E_z/c$	$F_{31} = -F_{13} = -B_y$	$F_{32} = -F_{23} = B_x$	$F_{33} = 0$

or, writing the "both indices up" form $F^{\mu\nu} = g^{\mu\alpha} F_{\alpha\beta}\, g^{\beta\nu}$ as a matrix

$$F^{\mu\nu} = \begin{bmatrix} \cdot & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & \cdot & -B_z & B_y \\ E_y/c & B_z & \cdot & -B_x \end{bmatrix}$$

$$\begin{cases} F^{0i} = -\dfrac{1}{c} E_i \\ F^{ij} = -\epsilon_{ijk} B_k \end{cases}$$

there's another rank-two tensor featuring $\vec{E}, \vec{B}$ that is useful, the "dual field-strength tensor"

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

where $\epsilon^{\mu\nu\alpha\beta}$ is the four-dimensional analogue of ϵ_{ijk} : $\epsilon^{0123} = 1$

swap any two indices $\rightarrow -1$ etc...

so that $$\tilde{F}^{\mu\nu} = \begin{bmatrix} \cdot & -B_x & -B_y & -B_z \\ B_x & \cdot & E_z/c & -E_y/c \\ B_y & -E_z/c & \cdot & E_x/c \\ B_z & E_y/c & -E_x/c & \cdot \end{bmatrix}$$

and the name "dual" indicates the transformation $\vec{E} \rightarrow c\vec{B}$, $\vec{B} \rightarrow -\vec{E}/c$ w.r.t $F^{\mu\nu}$.

Now consider Maxwell's equations :

- firstly the inhomogeneous (source carrying) equations : $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \quad = c\mu_0(c\rho)$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

consider $\partial_\mu F^{\mu\nu}$

$\nu=0$: $$\partial_\mu F^{\mu 0} = \frac{\partial}{\partial x}E_x/c + \frac{\partial}{\partial y}E_y/c + \frac{\partial}{\partial z}E_z/c = \frac{1}{c}\vec{\nabla}\cdot\vec{E} = \mu_0(c\rho) = \mu_0 J^0$$

$\nu=1$: $$\partial_\mu F^{\mu 1} = \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{-E_x}{c}\right) + \frac{\partial}{\partial y}B_z + \frac{\partial}{\partial z}(-B_y) = -\frac{1}{c^2}\frac{\partial}{\partial t}E_x + (\vec{\nabla}\times\vec{B})_x = \mu_0 J^1$$

etc...

indicating that we can write the inhomogeneous Maxwell's eqns in covariant form as $$\boxed{\partial_\mu F^{\mu\nu} = \mu_0 J^\nu}$$

- next the homogeneous (source free) equations : $\vec{\nabla}\cdot\vec{B} = 0$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

consider $\partial_\mu \tilde{F}^{\mu\nu}$

$\nu=0$: $$\partial_\mu \tilde{F}^{\mu 0} = \frac{\partial}{\partial x}B_x + \frac{\partial}{\partial y}B_y + \frac{\partial}{\partial z}B_z = \vec{\nabla}\cdot\vec{B}$$

$\nu=1$: $$\partial_\mu \tilde{F}^{\mu 1} = \frac{1}{c}\frac{\partial}{\partial t}(-B_x) + \frac{\partial}{\partial y}(-E_z/c) + \frac{\partial}{\partial z}(E_y/c) = -\frac{1}{c}\left(\frac{\partial B_x}{\partial t} + (\vec{\nabla}\times\vec{E})_x\right)$$

he relativistically correct equation of motion (analogous to $\frac{d}{dt} m\vec{u} = q(\vec{E} + \vec{u}\times\vec{B})$ non-relativistically) should be a covariant eqn which reduces to he non-rel equation for $|\vec{u}| \ll c$

we can infer its form: the invariant analogue of the time dependence is the proper time dependence and a Lorentz covariant featuring $\vec{u}$ is the four-velocity $U^\mu = \gamma(u)[c, \vec{u}]$,

so we might try $\frac{d}{d\tau}(mU^\mu)$ and we notice that the covariant $F^{\mu\nu}U_\nu$ will be linear in $\vec{E}, \vec{B}$ and $\vec{u}$

recall the definition of the four-momentum, $P^\mu = mU^\mu$ we suggest

$$\boxed{\frac{dP^\mu}{d\tau} = q F^{\mu\nu} U_\nu}$$

hen since $d\tau = \frac{dt}{\gamma(u)}$ we have for the space components: $\gamma(u)\frac{dp^i}{dt} = qF^{i\nu}U^\nu$

$$F^{i\nu}U_\nu = F^{i0}U^0 - F^{ij}U^j = \frac{1}{c}E_i\gamma(u)c - (-\epsilon_{ijk}B_k)\gamma(u)u^j$$

& dividing out the common factor of $\gamma(u)$: $\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{u}\times\vec{B})$

with $\vec{p} = \gamma(u)m\vec{u}$

& thus for $u \ll c$, we regain the non-rel result.

a cute check of consistency on this equation:

"dot" in with U_μ: $m U_\mu \frac{d}{d\tau}U^\mu = q U_\mu F^{\mu\nu} U^\nu$ but $F^{\mu\nu}$ is antisymmetric, so RHS $= 0$

$$\frac{1}{2}m\frac{d}{d\tau}(U_\mu U^\mu) \qquad U_\mu U^\mu = c^2 \quad \& \quad \frac{d}{d\tau}c^2 = 0 \quad \checkmark$$

Notice that our covariant approach supplies us with two Lorentz invariants (quantities that are the same in every inertial frame)

$$F_{\mu\nu}F^{\mu\nu}$$

$$= -2F^{0i}F^{0i} + F^{ij}F^{ij}$$

$$= -2\frac{1}{c^2}E_iE_i + \epsilon_{ijk}B_k\epsilon_{ij\bar{k}}B_{\bar{k}}$$

$$= -\frac{2}{c^2}|\vec{E}|^2 + 2\delta_{k\bar{k}}B_kB_{\bar{k}}$$

$$= -\frac{2}{c^2}\left(|\vec{E}|^2 - c^2|\vec{B}|^2\right)$$

$$F_{\mu\nu}\tilde{F}^{\mu\nu} \quad \left(= \epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}F^{\mu\nu}\right)$$

$$= -2F^{0i}\tilde{F}^{0i} + F^{ij}\tilde{F}^{ij}$$

$$= (-2)\left(-\frac{E_i}{c}\right)(-B_i) + \left(-\epsilon_{ijk}B_k\right)\left(\epsilon_{ij\bar{k}}E_{\bar{k}}/c\right)$$

$$= -\frac{2}{c}\vec{E}\cdot\vec{B} - \frac{2}{c}\vec{E}\cdot\vec{B}$$

$$= \frac{-4}{c}\vec{E}\cdot\vec{B}$$

[C]onservation laws & the stress-energy tensor

[C]onsider the force-density for a distribution of charge & current, $\vec{f} = \rho\vec{E} + \vec{J}\times\vec{B}$,
[a]nd note that we can form a four-vector

$$f^\mu = F^{\mu\nu} J_\nu$$

[w]hose space components $f^i = F^{i\nu}J_\nu = F^{i0}J^0 - F^{ij}J^j = \frac{1}{c}E_i c\rho - (-\epsilon_{ijk}B_k)J^j$

are the force density $\vec{f} = \rho\vec{E} + \vec{J}\times\vec{B}$

notice that the time component $f^0 = F^{0\nu}J_\nu = -F^{0i}J^i = \frac{1}{c}E_i J^i = \frac{1}{c}\vec{J}\cdot\vec{E}$
features the power density.

[W]e can eliminate the four-current from our equation using the covariant Maxwell's eqn

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$$

so $\mu_0 f^\mu = F^{\mu\nu}\partial^\alpha F_{\alpha\nu} = \partial^\alpha(F^{\mu\nu}F_{\alpha\nu}) - \underbrace{F_{\alpha\nu}\partial^\alpha F^{\mu\nu}}$

$$F_{\alpha\nu}\partial^\alpha F^{\mu\nu} = F_{\nu\alpha}\partial^\nu F^{\mu\alpha} \quad \begin{pmatrix}\alpha\to\nu\\ \nu\to\alpha\end{pmatrix}$$

$$= -F_{\alpha\nu}\partial^\nu F^{\mu\alpha} = F_{\alpha\nu}\partial^\nu F^{\alpha\mu}$$

$$= \partial^\alpha(F^{\mu\nu}F_{\alpha\nu}) - \frac{1}{2}F_{\alpha\nu}\left(\partial^\alpha F^{\mu\nu} + \partial^\nu F^{\alpha\mu}\right)$$

the homogeneous Maxwell eqns $0 = \partial_\mu \tilde{F}^{\mu\nu} = \partial_\mu \epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} = -\epsilon^{\nu\mu\alpha\beta}\partial_\mu F_{\alpha\beta}$

which can be written $0 = \partial^\mu F^{\alpha\beta} + \partial^\alpha F^{\beta\mu} + \partial^\beta F^{\mu\alpha}$ [check]

then $\mu_0 f^\mu = \partial^\alpha(F^{\mu\nu}F_{\alpha\nu}) - \frac{1}{2}F_{\alpha\nu}\left(-\partial^\mu F^{\nu\alpha}\right)$

$$= \partial^\alpha(F^{\mu\nu}F_{\alpha\nu}) - \frac{1}{2}F_{\nu\alpha}\partial^\mu F^{\nu\alpha} = \partial^\alpha(F^{\mu\nu}F_{\alpha\nu}) - \frac{1}{4}\partial^\mu(F_{\nu\alpha}F^{\nu\alpha})$$

now define $\boxed{\Theta^{\alpha\beta} = \frac{1}{\mu_0}\left[F^\alpha{}_\lambda F^{\lambda\beta} + \frac{1}{4}g^{\alpha\beta}F_{\mu\lambda}F^{\mu\lambda}\right]}$

then $\partial_\alpha\Theta^{\alpha\beta} = \frac{1}{\mu_0}\left[\partial_\alpha(F^\alpha{}_\lambda F^{\lambda\beta}) + \frac{1}{4}\partial^\beta(F_{\mu\lambda}F^{\mu\lambda})\right]$

so that $\underline{\partial_\alpha \Theta^{\alpha\mu} = -f^\mu}$

what's the content of the "stress-energy" tensor $\Theta^{\alpha\beta} = \frac{1}{\mu_0}\left[F^\alpha{}_\lambda F^{\lambda\beta} + \frac{1}{4} g^{\alpha\beta} F_{\mu\lambda}F^{\mu\lambda}\right]$?

$$\Theta^{00} = \frac{1}{\mu_0}\left[+F^{0i}F^{0i} + \frac{1}{4}\left(-\frac{2}{c^2}\right)\left(E^2 - c^2B^2\right)\right] = \frac{1}{\mu_0}\left[+\frac{E^2}{c^2} - \frac{1}{2}\frac{E^2}{c^2} + \frac{1}{2}B^2\right] = \frac{1}{2\mu_0 c^2}\left[E^2 + c^2B^2\right]$$

$$= \frac{1}{2}\varepsilon_0\left(E^2 + c^2B^2\right) = \underline{u_{em}}$$

$$\Theta^{0i} = \frac{1}{\mu_0}\left[-F^{0j}F^{ji}\right] = \frac{1}{\mu_0}\frac{1}{c}E_j(\epsilon_{jik}B_k) = \frac{1}{c}\cdot\frac{1}{\mu_0}\epsilon_{ijk}E_jB_k = \frac{1}{c}\left(\vec{E}\times\vec{B}/\mu_0\right)_i = S_i/c$$

$$\Theta^{ij} = \frac{1}{\mu_0}\left[F^i{}_\lambda F^{\lambda j} + \frac{1}{4}(-\delta_{ij})\left(-\frac{2}{c^2}\right)\left(E^2 - c^2B^2\right)\right]$$

$$= \frac{1}{\mu_0}\left[F^{i0}F^{0j} - F^{ik}F^{kj} + \frac{1}{2}\delta_{ij}\frac{1}{c^2}\left(E^2 - c^2B^2\right)\right]$$

$$= \frac{1}{\mu_0}\left[-\frac{E_i}{c}\frac{E_j}{c} - \left(-\epsilon_{ik\ell}B_\ell\right)\left(-\epsilon_{kjm}B_m\right) + \frac{1}{2}\delta_{ij}\frac{1}{c^2}\left(E^2 - c^2B^2\right)\right]$$

$$= \frac{1}{\mu_0 c^2}\left[-E_iE_j - c^2\underbrace{\epsilon_{k\ell i}\epsilon_{kjm}}_{\delta_{\ell j}\delta_{im} - \delta_{\ell m}\delta_{ij}}B_\ell B_m + \frac{1}{2}\delta_{ij}\left(E^2 - c^2B^2\right)\right]$$

$$= \varepsilon_0\left[-E_iE_j - c^2B_iB_j + c^2\delta_{ij}B^2 + \frac{1}{2}\delta_{ij}\left(E^2 - c^2B^2\right)\right]$$

$$= -\varepsilon_0\left[E_iE_j + c^2B_iB_j - \frac{1}{2}\delta_{ij}\left(E^2 + c^2B^2\right)\right] = -T_{ij}$$

↑ the Maxwell stress tensor

$$\Theta^{\mu\nu} = \begin{bmatrix} u_{em} & \vec{S}/c \\ \vec{S}/c & -\overleftrightarrow{T} \end{bmatrix}$$

and Θ is symmetric $\Theta^{\mu\nu} = \Theta^{\nu\mu}$

and traceless $\Theta_\mu{}^\mu = 0$.

We can show that $\partial_\alpha \Theta^{\alpha\mu} + f^\mu = 0$ contains two conservation laws in a single covariant expression:

$\mu = 0$: $0 = \partial_\alpha \Theta^{\alpha 0} + f^0 = \partial_0 u_{em} + \partial_i (\vec{S})_i/c + \frac{1}{c}\vec{J}\cdot\vec{E}$

$$\Rightarrow \quad 0 = \frac{\partial u_{em}}{\partial t} + \vec{\nabla}\cdot\vec{S} + \vec{J}\cdot\vec{E}$$ "conservation of energy"

$\mu = i$: $0 = \partial_\alpha \Theta^{\alpha i} + f^i = \partial_0 (\vec{S})_i/c + \partial_j(-T_{ij}) + F_i$

$$\Rightarrow \quad 0 = \frac{\partial}{\partial t}\left(\frac{\vec{S}}{c^2}\right) - \vec{\nabla}\cdot\overleftrightarrow{T} + \vec{F}$$ "conservation of linear momentum"