WAVEGUIDES & CAVITIES

We now return to the problem of electromagnetic fields in the presence of metallic boundaries. We will be particularly interested in wave-like solutions in open-ended ("waveguide") and closed ("cavity") pipes.

First let's recall the behavior of fields at the surface of and within a conductor:

PERFECT CONDUCTOR: surface charge density, Σ and surface current density, $\vec{K}$ ensure $\vec{E}_c$ & $\vec{H}_c$ inside the conductor are zero.

for outward facing normal $\hat{n}$:

$$\hat{n}\cdot\vec{D} = \Sigma \qquad \hat{n}\cdot\vec{B} = 0$$
$$\hat{n}\times\vec{E} = 0 \qquad \hat{n}\times\vec{H} = \vec{K}$$

so the $\vec{E}$-field just outside a perfect conductor is normal and the $\vec{H}$-field is tangential.

REAL (GOOD) CONDUCTOR: fields are not zero inside the conductor but rather attenuate rapidly to zero over a distance δ, the "skin depth"

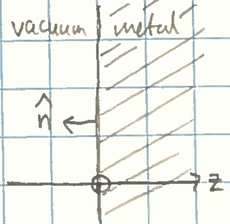
Inside the conductor: $\vec{J} = \sigma\vec{E}$ "Ohm's law"

at the boundary $\hat{n}\times(\vec{H}-\vec{H}_c) = 0$

i.e. continuity of tangential $\vec{H}$.

inside the conductor $\vec{\nabla}\times\vec{H}_c = \sigma\vec{E}_c + \dfrac{\partial\vec{D}_c}{\partial t} \xrightarrow[\text{fields}]{\text{harmonic}} (\sigma - i\omega\varepsilon)\vec{E}_c \xrightarrow[\text{conductor}]{\text{good}} \sigma\vec{E}_c$

$$\vec{\nabla}\times\vec{E}_c = -\frac{\partial\vec{B}_c}{\partial t} \longrightarrow i\omega\mu\vec{H}_c$$



so $\vec{H}_c = \dfrac{-i}{\mu\omega}\vec{\nabla}\times\vec{E}_c$

$$\vec{E}_c \simeq \frac{1}{\sigma}\vec{\nabla}\times\vec{H}_c$$

anticipating the rapid attenuation in the z direction, we'll neglect derivatives in directions other than z, $\vec{\nabla} \simeq \hat{z}\dfrac{\partial}{\partial z} = -\hat{n}\dfrac{\partial}{\partial z}$

& $\vec{H}_c \simeq \dfrac{i}{\mu\omega}\hat{n}\times\dfrac{\partial\vec{E}_c}{\partial z}$ & $\vec{E}_c \simeq -\dfrac{1}{\sigma}\hat{n}\times\dfrac{\partial\vec{H}_c}{\partial z}$

thus $\hat{n}\cdot\vec{H}_c \simeq 0$
$\hat{n}\cdot\vec{E}_c \simeq 0$

$$\hat{n}\times\vec{H}_c \simeq \frac{i}{\mu\omega}\frac{\partial}{\partial z}\hat{n}\times(\hat{n}\times\vec{E}_c) \simeq \frac{i}{\mu\omega}\frac{\partial}{\partial z}\left[\hat{n}(\cancelto{0}{\hat{n}\cdot\vec{E}_c}) - \vec{E}_c\right] \simeq \frac{i}{\mu\omega\sigma}\frac{\partial^2}{\partial z^2}\hat{n}\times\vec{H}_c$$

$$\frac{\partial^2}{\partial z^2}(\hat{n}\times\vec{H}_c) + i\mu\omega\sigma\,(\hat{n}\times\vec{H}_c) \simeq 0$$

recalling the skin depth $\delta \equiv \sqrt{\frac{2}{\mu\omega\sigma}}$

$$\left(\frac{\partial^2}{\partial z^2} + \frac{2i}{\delta^2}\right)\hat{n}\times\vec{H}_c = 0 \quad\Rightarrow\quad \vec{H}_c \simeq \vec{H}_{\parallel}\, e^{-z/\delta} e^{iz/\delta}$$

with $\vec{H}_{\parallel}$ the field just outside the conductor

then since $\vec{E}_c \simeq -\frac{1}{\sigma}\frac{\partial}{\partial z}\hat{n}\times\vec{H}_c \simeq \sqrt{\frac{\mu\omega}{2\sigma}}(1-i)(\hat{n}\times\vec{H}_{\parallel})\, e^{-z/\delta} e^{iz/\delta}$

N.B orthogonal to $\hat{z}$ and to $\vec{H}_{\parallel}$

N.B $\frac{|\vec{H}_c|}{|\vec{E}_c|} \sim \sqrt{\frac{\mu\omega}{\sigma}} \ll 1$ for a good conductor.

but now note that $\hat{n}\times(\vec{E}-\vec{E}_c) = 0$, which should hold at the boundary of the conductor implies that just outside the conductor the $\vec{E}$ field has a tangential component

$$\vec{E}_{\parallel} \simeq \sqrt{\frac{\mu\omega}{2\sigma}}(1-i)(\hat{n}\times\vec{H}_{\parallel})$$

(as expected as $\sigma\to\infty$ "perfect conductor" this tends to zero)

there is a correspondingly small normal component of $\vec{B}$ outside the conductor.

The tangential $\vec{E}$ outside the conductor, not present for a perfect conductor leads to a net flow of energy into the conductor:

time average, $\langle\vec{S}\rangle = \frac{1}{2}\mathrm{Re}(\vec{E}\times\vec{H}^*)$

flow INTO conductor $(-\hat{n})\cdot\langle\vec{S}\rangle = -\frac{1}{2}\mathrm{Re}(\hat{n}\cdot\vec{E}\times\vec{H}^*) = -\frac{1}{2}\mathrm{Re}\left[\sqrt{\frac{\mu\omega}{2\sigma}}(1-i)\,\hat{n}\cdot(\hat{n}\times\vec{H}_{\parallel})\times\vec{H}_{\parallel}^*\right]$

$$= \frac{\mu\omega\delta}{4}|\vec{H}_{\parallel}|^2$$

this energy is dissipated as ohmic heating in a thin layer of conductor just below the surface

$$\vec{J} = \sigma \vec{E}_c \simeq \frac{1}{\delta}(1-i)(\hat{n}\times\vec{H}_{\parallel})\, e^{-z/\delta} e^{iz/\delta}$$

(time averaged)
rate of energy loss per unit volume $= \frac{1}{2}\vec{J}\cdot\vec{E}^* = \frac{1}{2}\frac{1}{\sigma}|\vec{J}|^2$

$$\int_0^\infty dz\, \frac{1}{2}\frac{1}{\sigma}|J|^2 = \frac{1}{2\sigma}\cdot\frac{1}{\delta^2}\, 2|\vec{H}_{\parallel}|^2 \int_0^\infty dz\, e^{-2z/\delta} = \frac{\mu\omega}{2}|\vec{H}_{\parallel}|^2\cdot\frac{\delta}{2} = \frac{\mu\omega\delta}{4}|\vec{H}_{\parallel}|^2$$

which is the energy flow we computed previously.

we can approximate the effect of $\vec{J}$ outside the conductor by treating it as an EFFECTIVE SURFACE CURRENT

$$\vec{K}_{eff} = \int_0^\infty \vec{J}\, dz = \hat{n}\times\vec{H}_{\parallel}$$

and it follows that the power loss per unit area of surface is

$$\frac{dP}{dA} = \frac{\mu\omega\delta}{4}|\vec{K}_{eff}|^2 = \frac{1}{2}\cdot\frac{1}{\sigma\delta}|\vec{K}_{eff}|^2$$

by analogy with $P = \frac{1}{2}RI^2$ (for the time average of AC currents) we can call $\frac{1}{\sigma\delta}$ the "surface resistance" of the conductor

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for good conductors acting as waveguides or cavities we can follow an approximation procedure where we

1. solve for $\vec{E}, \vec{B}$ in the cavity assuming perfectly conducting boundaries
2. find the $\vec{K}_{eff}$ that allows the fields at the boundary
3. compute the approximate resistive losses.

Cylindrical cavities & waveguides

Consider a hollow metal cylinder of arbitrary cross sectional shape. For the z-axis we'll choose the axis of the cylinder. We'll allow the cylinder to contain a (non-dispersive) simple material (constant ε, μ).

Monochromatic fields ($\sim e^{-i\omega t}$) satisfy

$$\vec{\nabla}\times\vec{E} = i\omega\mu\vec{H} \qquad \vec{\nabla}\cdot\vec{E}=0$$

$$\vec{\nabla}\times\vec{H} = -i\omega\varepsilon\vec{E} \qquad \vec{\nabla}\cdot\vec{H}=0$$

and thus $\left(\nabla^2 + \mu\varepsilon\omega^2\right)\left\{\begin{matrix}\vec{E}\\ \vec{H}\end{matrix}\right\} = 0$ (Helmholtz)

Let's assume sinusoidal behavior in the z-direction with wavenumber k $\sim e^{\pm ikz}$

$$\text{e.g.}\quad \vec{E}(\vec{r},t) = \vec{E}(x,y)\,e^{\pm ikz - i\omega t}$$

then $\left(\nabla_\perp^2 + \mu\varepsilon\omega^2 - k^2\right)\vec{E}(x,y) = 0$ & similarly for $\vec{H}$

with $\nabla_\perp^2 = \nabla^2 - \frac{\partial^2}{\partial z^2}$

Separating $\vec{E}$ into components along $\hat{z}$ and perpendicular to $\hat{z}$, $\vec{E} = \hat{z}E_z + \vec{E}_\perp$,

(where $\vec{E}_\perp = \underline{(\hat{z}\times\vec{E})\times\hat{z}}$)

$$\left\{\begin{aligned} &-\hat{z}\times(\hat{z}\times\vec{E})\\ &= -\hat{z}E_z + \vec{E}\\ &= \vec{E} - \hat{z}E_z \ \checkmark\end{aligned}\right.$$

and defining $\vec{\nabla}_\perp = \vec{\nabla} - \hat{z}\frac{\partial}{\partial z}$ we can write

$$\vec{\nabla}\times\vec{E} = \left(\vec{\nabla}_\perp + \hat{z}\frac{\partial}{\partial z}\right)\times\left(\hat{z}E_z + \vec{E}_\perp\right) = \vec{\nabla}_\perp\times(\hat{z}E_z) + \vec{\nabla}_\perp\times\vec{E}_\perp + \frac{\partial}{\partial z}\hat{z}\times\vec{E}_\perp$$

and $\hat{z}\cdot\vec{\nabla}\times\vec{E} = 0 + \hat{z}\cdot\vec{\nabla}_\perp\times\vec{E}_\perp + 0$

$$\hat{z}\times(\vec{\nabla}\times\vec{E}) = \hat{z}\times\left(-\hat{z}\times\vec{\nabla}_\perp E_z\right) + 0 + \frac{\partial}{\partial z}\hat{z}\times(\hat{z}\times\vec{E}_\perp)$$

$$= \vec{\nabla}_\perp E_z - \frac{\partial}{\partial z}\vec{E}_\perp$$

$$\frac{\partial \vec{E}_\perp}{\partial z} + i\omega\mu\, \hat{z}\times\vec{H}_\perp = \vec{\nabla}_\perp E_z \quad , \quad \hat{z}\cdot(\vec{\nabla}_\perp \times \vec{E}_\perp) = i\omega\mu H_z \quad , \quad \vec{\nabla}_\perp\cdot\vec{E}_\perp = -\frac{\partial E_z}{\partial z}$$

$$\frac{\partial \vec{H}_\perp}{\partial z} - i\omega\epsilon\, \hat{z}\times\vec{E}_\perp = \vec{\nabla}_\perp H_z \quad , \quad \hat{z}\cdot(\vec{\nabla}_\perp \times \vec{H}_\perp) = -i\omega\epsilon E_z \quad , \quad \vec{\nabla}_\perp\cdot\vec{H}_\perp = -\frac{\partial H_z}{\partial z}$$

if $\vec{E},\vec{H} \sim e^{ikz}$ the leftmost equations become

$$ik\vec{E}_\perp + i\omega\mu\, \hat{z}\times\vec{H}_\perp = \vec{\nabla}_\perp E_z$$

$$ik\vec{H}_\perp - i\omega\epsilon\, \hat{z}\times\vec{E}_\perp = \vec{\nabla}_\perp H_z \quad \longrightarrow \quad ik\, \hat{z}\times\vec{H}_\perp + i\omega\epsilon\, \vec{E}_\perp = \hat{z}\times\vec{\nabla}_\perp H_z$$

$$\& \quad ik\vec{E}_\perp + i\omega\mu\left(\frac{1}{ik}\right)\left(\hat{z}\times\vec{\nabla}_\perp H_z - i\omega\epsilon E_\perp\right) = \vec{\nabla}_\perp E_z$$

$$-k^2\vec{E}_\perp + \omega^2\mu\epsilon\vec{E}_\perp + i\omega\mu\, \hat{z}\times\vec{\nabla}_\perp H_z = ik\vec{\nabla}_\perp E_z$$

$$\vec{E}_\perp = \frac{i}{\mu\epsilon\omega^2 - k^2}\left[k\,\vec{\nabla}_\perp E_z - \omega\mu\, \hat{z}\times\vec{\nabla}_\perp H_z\right]$$

and similarly $$\vec{H}_\perp = \frac{i}{\mu\epsilon\omega^2 - k^2}\left[k\,\vec{\nabla}_\perp H_z + \omega\epsilon\, \hat{z}\times\vec{\nabla}_\perp E_z\right]$$

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e special case of "TEM" modes:

the solutions above require at least one of E_z or H_z to be nonzero, but there are solutions to [EM] with $E_z = H_z = 0$, then $\vec{\nabla}_\perp\times\vec{E}_\perp = 0$ & $\vec{\nabla}_\perp\cdot\vec{E}_\perp = 0$ and $k^2 = \epsilon\mu\omega^2$

and $$\vec{H}_\perp = \frac{\omega\epsilon}{k}\hat{z}\times\vec{E}_\perp = \sqrt{\frac{\epsilon}{\mu}}\,\hat{z}\times\vec{E}_\perp$$

which looks a lot like plane waves in an infinite medium.

In fact no such solution is possible in a single hollow cylindrical conductor

$\vec{\nabla}_\perp\times\vec{E}_\perp = 0 \Rightarrow \vec{E}_\perp = -\vec{\nabla}_\perp\varphi$ & $\vec{\nabla}_\perp\cdot\vec{E}_\perp = 0 \Rightarrow \nabla_\perp^2\varphi = 0$

since φ is constant on the conducting boundary, the only solution to $\nabla_\perp^2\varphi = 0$ is $\varphi = $ const which gives $\vec{E}_\perp = \vec{0}$!

for a perfect conductor the boundary conditions are $\hat{n}\times\vec{E}\big|_s = 0$
& $\hat{n}\cdot\vec{H}\big|_s = 0$

These boundary conditions are satisfied by two classes of solution:

→ TM modes "transverse magnetic waves"

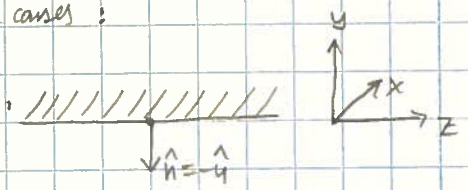
$H_z = 0$ everywhere, $E_z\big|_s = 0$

→ TE modes "transverse electric waves"

$E_z = 0$ everywhere, $\left.\dfrac{\partial H_z}{\partial n}\right|_s = 0$

we can easily check that $\hat{n}\times\vec{E}\big|_s = 0$ in both these cases:

consider a small section of conducting surface



$$\hat{n}\times\vec{E} = \hat{n}\times(E_z\hat{z} + \vec{E}_\perp) = E_z\,\hat{n}\times\hat{z} + \hat{n}\times\vec{E}_\perp$$

→ TM modes $\hat{n}\times\vec{E}\big|_s = \hat{n}\times\vec{E}_\perp$ but $\vec{E}_\perp = \dfrac{ik}{\mu\epsilon\omega^2 - k^2}\vec{\nabla}_\perp E_z$

so $\hat{n}\times\vec{E}\big|_s = \dfrac{ik}{\mu\epsilon\omega^2-k^2}\,\hat{n}\times\vec{\nabla}_\perp E_z$

$$\hat{n}\times\vec{\nabla}_\perp = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & -1 & 0 \\ \partial/\partial x & \partial/\partial y & 0 \end{vmatrix} = \hat{z}\frac{\partial}{\partial x}$$

$= \dfrac{ik}{\mu\epsilon\omega^2-k^2}\,\hat{z}\left.\dfrac{\partial E_z}{\partial x}\right|_s$

but $E_z = 0$ everywhere on the surface $\Rightarrow \left.\dfrac{\partial E_z}{\partial x}\right|_s = 0$

$\hat{n}\times\vec{E}\big|_s = 0$ ✓

→ TE modes $\vec{E}_\perp = \dfrac{-i\omega\mu}{\mu\epsilon\omega^2-k^2}\,\hat{z}\times\vec{\nabla}_\perp H_z$ $\Rightarrow \hat{n}\times\vec{E}_\perp = \dfrac{-i\omega\mu}{\mu\epsilon\omega^2-k^2}(-\hat{y})\times(\hat{z}\times\vec{\nabla}_\perp)H_z$

$= \dfrac{i\omega\mu}{\mu\epsilon\omega^2-k^2}\,\hat{z}\,\dfrac{\partial H_z}{\partial y}$

$\Rightarrow \hat{n}\times\vec{E}\big|_s = 0$ if $\left.\dfrac{\partial H_z}{\partial n}\right|_s = 0$ ✓

$\boxed{\text{TM}}$

$H_z = 0 \qquad E_z|_S = 0$

$$(\pm\hat{z})\times\vec{E}_\perp = Z_{TM}\vec{H}_\perp \quad \Big| \quad Z_{TM} = \frac{k}{\varepsilon\omega} = \frac{k}{k_0}\sqrt{\frac{\mu}{\varepsilon}}$$

$$\vec{E}_\perp = \pm\frac{ik}{\gamma^2}\vec{\nabla}_\perp \psi_{TM} e^{\pm ikz} \quad \Big| \quad \psi_{TM} e^{\pm ikz} = E_z$$

$$\psi_{TM}|_S = 0$$

$\boxed{\text{TE}}$

$E_z = 0 \qquad \frac{\partial H_z}{\partial n}\Big|_S = 0$

$$(\pm\hat{z})\times\vec{E}_\perp = Z_{TE}\vec{H}_\perp \quad \Big| \quad Z_{TE} = \frac{\mu\omega}{k} = \frac{k_0}{k}\sqrt{\frac{\mu}{\varepsilon}}$$

$$\vec{H}_\perp = \pm\frac{ik}{\gamma^2}\vec{\nabla}_\perp \psi_{TE} e^{\pm ikz} \quad \Big| \quad \psi_{TE} e^{\pm ikz} = H_z$$

$$\frac{\partial \psi_{TE}}{\partial n}\Big|_S = 0$$

$$(\nabla_\perp^2 + \gamma^2)\psi = 0 \quad , \quad \gamma^2 = \mu\varepsilon\omega^2 - k^2$$

These equations (the 2D Helmholtz equation & Dirichlet or Neumann boundary conditions) define an eigenvalue system.

Oscillatory solutions will correspond to γ real – applying the boundary conditions will restrict the possible values of γ^2:

$$\nabla_\perp^2 \psi_\lambda = -\gamma_\lambda^2 \psi_\lambda$$

γ_λ^2 – eigenvalues
ψ_λ – eigenfunctions

$\lambda = 1, 2, 3 \ldots$

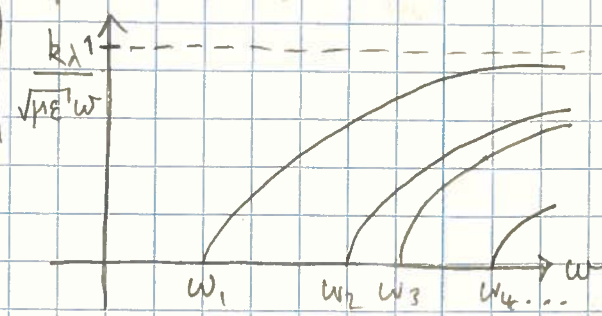
the corresponding wavenumbers are $k_\lambda^2 = \mu\varepsilon\omega^2 - \gamma_\lambda^2$

and clearly if $\gamma_\lambda > \sqrt{\mu\varepsilon}\,\omega$, k_λ becomes imaginary and the solutions are non-propagating.

we can define a "cutoff frequency", $\omega_\lambda \equiv \gamma_\lambda / \sqrt{\mu\varepsilon}$

and in terms of this $\quad k_\lambda = \sqrt{\mu\varepsilon}\sqrt{\omega^2 - \omega_\lambda^2}$

$$\frac{k_\lambda}{\sqrt{\mu\varepsilon}\,\omega} = \sqrt{1 - \left(\frac{\omega_\lambda}{\omega}\right)^2}$$



ORTHOGONALITY OF MODES

We can construct any field configuration in the waveguide consistent with the boundary conditions as a superposition of the TE and TM eigenmodes. The orthogonality of the modes makes this relatively simple.

$$-\gamma_p^2 \psi_p = \nabla_\perp^2 \psi_p \quad \Rightarrow \quad -\gamma_p^2 \int dA\, \psi_q \psi_p = \int dA\, \psi_q \nabla_\perp^2 \psi_p$$

$$-\gamma_q^2 \psi_q = \nabla_\perp^2 \psi_q \quad \Rightarrow \quad -\gamma_q^2 \int dA\, \psi_q \psi_p = \int dA\, \psi_p \nabla_\perp^2 \psi_q$$

and subtracting

$$(\gamma_q^2 - \gamma_p^2) \int dA\, \psi_q \psi_p = \int dA \left(\psi_q \nabla_\perp^2 \psi_p - \psi_p \nabla_\perp^2 \psi_q \right)$$

by Green's second identity $\int dA \left(\psi_q \nabla_\perp^2 \psi_p - \psi_p \nabla_\perp^2 \psi_q \right)$

$$= \oint dl \left(\psi_q \frac{\partial \psi_p}{\partial n} - \psi_p \frac{\partial \psi_q}{\partial n} \right).$$

but this last integral is zero by the boundary conditions for both TE & TM waves

$$\Rightarrow \quad (\gamma_q^2 - \gamma_p^2) \int dA\, \psi_q \psi_p = 0 \quad \& \quad \int dA\, \psi_q \psi_p = 0 \text{ if } \gamma_p \neq \gamma_q$$

For TE modes $\quad \vec{E}_p = -Z_p^{TE}\, \hat{z} \times \vec{H}_p \quad / \quad \vec{H}_p = \hat{z} H_z + \vec{H}_\perp$

$$= \hat{z} \psi_p + \frac{i k_p}{\gamma_p^2} \vec{\nabla}_\perp \psi_p$$

$$\text{so } \int dA\, \vec{H}_p \cdot \vec{H}_q = \frac{1}{\gamma_p^2} \frac{1}{\gamma_q^2} \int dA \left(i k_p \vec{\nabla}_\perp \psi_p + \gamma_p^2 \psi_p \hat{z} \right) \cdot \left(i k_q \vec{\nabla}_\perp \psi_q + \gamma_q^2 \psi_q \hat{z} \right)$$

$$= \frac{1}{\gamma_p^2} \frac{1}{\gamma_q^2} \int dA \left(-k_p k_q\, \vec{\nabla}_\perp \psi_p \cdot \vec{\nabla}_\perp \psi_q + \gamma_p^2 \gamma_q^2 \psi_p \psi_q \right)$$

$$= \int dA\, \psi_p \psi_q - \frac{k_p k_q}{\gamma_p^2 \gamma_q^2} \left(\int dA\, \vec{\nabla}_\perp \cdot \left(\psi_p \vec{\nabla}_\perp \psi_q \right) - \int dA\, \psi_p \underbrace{\nabla_\perp^2 \psi_q}_{-\gamma_q^2 \psi_q} \right)$$

$$= \left(1 + \frac{k_p k_q}{\gamma_p^2} \right) \int dA\, \psi_p \psi_q - \frac{k_p k_q}{\gamma_p^2 \gamma_q^2} \oint dl\, \psi_p \frac{\partial \psi_q}{\partial n}$$

$$= 0 \text{ if } p \neq q.$$

& similarly for $\int dA\, \vec{E}_p \cdot \vec{E}_q$ & for TM modes.

ENERGY FLOW IN A WAVEGUIDE

A useful measure of the rate of transmission down a waveguide is the "energy velocity"

$$V_E = \frac{\langle P\rangle}{\langle U\rangle/L} = \frac{\text{time-averaged power through a cross-section}}{\text{time averaged energy per unit length}}$$

$$\langle P\rangle = \int dA\, \hat{z}\cdot\langle\vec{S}\rangle \qquad \hat{z}\cdot\langle\vec{S}\rangle = \frac{1}{2}\mathrm{Re}\left(\vec{E}\times\vec{H}^*\right)\cdot\hat{z}$$

so only the transverse parts of $\vec{E}$, $\vec{H}$ contribute

$$= \frac{1}{2}\mathrm{Re}\left(\vec{E}_\perp\times\vec{H}_\perp^*\right)\cdot\hat{z} = \frac{1}{2}\mathrm{Re}\left(\hat{z}\times\vec{E}_\perp\cdot\vec{H}_\perp^*\right)$$

e.g. a TE wave has $\hat{z}\cdot\langle\vec{S}\rangle = \frac{1}{2}\mathrm{Re}\left(Z_{TE}|\vec{H}_\perp|^2\right) = \frac{1}{2}\mu\frac{\omega}{k}|\vec{H}_\perp|^2$

$$\& \langle P_{TE}\rangle = \frac{1}{2}\mu\frac{\omega}{k}\int dA\, |\vec{H}_\perp|^2 = \frac{1}{2}\mu\frac{\omega}{k}\cdot\frac{k^2}{\gamma^4}\int dA\, |\vec{\nabla}_\perp\psi|^2$$

$$= \frac{\mu\omega k}{2\gamma^4}\int dA\,(\vec{\nabla}_\perp\psi^*)\cdot(\vec{\nabla}_\perp\psi) = \frac{\mu\omega k}{2\gamma^4}\int dA\left[\vec{\nabla}_\perp\cdot(\psi^*\vec{\nabla}_\perp\psi) - \psi^*\nabla_\perp^2\psi\right]$$

$$= \frac{\mu\omega k}{2\gamma^4}\oint dl\, \psi^*\frac{\partial\psi}{\partial n} + \frac{\mu\omega k}{2\gamma^2}\int dA\,|\psi|^2 = \frac{\mu\omega k}{2\gamma^2}\int dA\,|H_z|^2$$

[show this!]

$$\text{now } \langle U\rangle = \underset{\text{time-av}}{\frac{1}{2}}\cdot L\int dA\,\frac{1}{2}\left(\varepsilon|\vec{E}|^2 + \mu|\vec{H}|^2\right)$$

$$\Rightarrow \langle U_{TE}\rangle = \frac{L}{4}\int dA\left(\varepsilon|E_\perp|^2 + \mu|\vec{H}_\perp + \hat{z}H_z|^2\right)$$

$$= L/4\int dA\left(\varepsilon Z_{TE}^2|H_\perp|^2 + \mu\left|\frac{ik}{\gamma^2}\vec{\nabla}_\perp H_z + \hat{z}H_z\right|^2\right)$$

$$= L/4\int dA\left(\varepsilon Z_{TE}^2\frac{k^2}{\gamma^4}|\vec{\nabla}_\perp H_z|^2 + \mu\frac{k^2}{\gamma^4}|\vec{\nabla}_\perp H_z|^2 + \mu|H_z|^2\right)$$

$$= L/4\left[\left(\varepsilon Z_{TE}^2\frac{k^2}{\gamma^4} + \mu\frac{k^2}{\gamma^4}\right)\int dA\,|\vec{\nabla}_\perp H_z|^2 + \mu\int dA\,|H_z|^2\right]$$

$$\langle U_{TE}\rangle = L/4\left[\left(\epsilon \frac{\mu^2\omega^2}{\cancel{k^2}}\frac{\cancel{k^2}}{\gamma^4} + \frac{\mu k^2}{\gamma^4}\right)\cdot \gamma^2 + \mu\right]\int dA\, |H_z|^2$$

$$= L/4\left[\frac{\mu}{\gamma^2}\left(\epsilon\mu\omega^2 + k^2\right) + \mu\right]\int dA\, |H_z|^2$$

$$= \frac{L}{4}\frac{\mu}{\gamma^2}\left[\epsilon\mu\omega^2 + k^2 + \epsilon\mu\omega^2 - k^2\right]\int dA\, |H_z|^2$$

$$\langle U_{TE}\rangle / L = \frac{\epsilon\mu^2\omega^2}{2\gamma^2}\int dA\, |H_z|^2$$

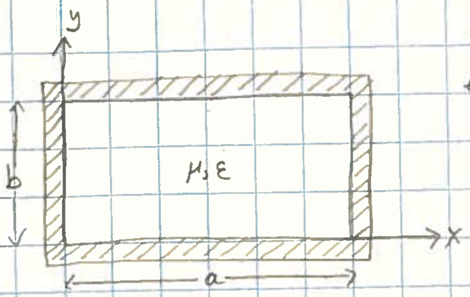
& finally $v_E = \frac{\mu\omega k}{\cancel{2\gamma^2}}\cdot\frac{\cancel{2\gamma^2}}{\epsilon\mu^2\omega^2} = \frac{k}{\epsilon\mu\omega}$

since $\gamma^2 = \mu\varepsilon\omega^2 - k^2$

$0 = 2\mu\varepsilon\omega\frac{d\omega}{dk} - 2k \quad \Rightarrow \frac{d\omega}{dk} = \frac{k}{\epsilon\mu\omega} = v_g \qquad \& \ \underline{v_E = v_g}$

energy flows down the waveguide at the group velocity.

modes in a rectangular waveguide



e.g. TE modes $(\psi = H_z)$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2\right)\psi = 0$$

$$\left.\frac{\partial\psi}{\partial x}\right|_{x=0} = 0 \Big/ \left.\frac{\partial\psi}{\partial x}\right|_{x=a} = 0$$

$$\left.\frac{\partial\psi}{\partial y}\right|_{y=0} = 0 \Big/ \left.\frac{\partial\psi}{\partial y}\right|_{y=b} = 0$$

sin & cos solutions – sin solutions will not satisfy these boundary conditions

$$\Rightarrow \psi(x,y) = A\cos\alpha x \cdot \cos\beta y \qquad \text{with } \gamma^2 = \alpha^2 + \beta^2$$

$$0 = \left.\frac{\partial\psi}{\partial x}\right|_{x=a} = (A\cos\beta y)(-\alpha\sin\alpha a) \qquad \Rightarrow \alpha = m\pi/a$$

$$0 = \left.\frac{\partial\psi}{\partial y}\right|_{y=b} = (A\cos\alpha x)(-\beta\sin\beta b) \qquad \Rightarrow \beta = n\pi/b$$

eigenfunctions

$$\Rightarrow \psi_{mn}(x,y) = H_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

eigenvalues

$$\gamma_{mn}^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

odes in a cylindrical cavity

ow consider the case where our hollow cylindrical conductor has conducting end planes at $z=0$ and $z=d$.

In this case we require standing waves, $e^{\pm ikz} \to (\cos kz, \sin kz)$

and the boundary conditions will restrict k to $p \cdot \pi/d$ for $p = 0, 1, 2 \ldots$

$\boxed{TM}$
$$E_z = \tfrac{1}{2}(\psi e^{ikz} + \psi e^{-ikz}) = \psi(x,y)\cos kz = \psi(x,y)\cos\frac{p\pi z}{d}$$
$$\vec{E}_\perp = \tfrac{1}{2}\left(+\frac{ik}{\gamma^2}\vec{\nabla}_\perp\psi\, e^{ikz} + \left(\frac{-ik}{\gamma^2}\right)\vec{\nabla}_\perp\psi e^{-ikz}\right) = -\frac{k}{\gamma^2}\vec{\nabla}_\perp\psi \sin kz = -\frac{p\pi}{d}\frac{1}{\gamma^2}\vec{\nabla}_\perp\psi \sin\frac{p\pi z}{d}$$

so $\vec{E}_\perp(z=0,d) = 0$ as required by $\hat{n}\times\vec{E}|_S = 0$

$\boxed{TE}$
$$H_z = \frac{1}{2i}(\psi e^{ikz} - \psi e^{-ikz}) = \psi(x,y)\sin kz = \psi(x,y)\sin\frac{p\pi z}{d}$$
$$\vec{H}_\perp = \frac{1}{2i}\left(\frac{ik}{\gamma^2}\vec{\nabla}_\perp\psi\, e^{ikz} - \left(\frac{-ik}{\gamma^2}\right)\vec{\nabla}_\perp\psi e^{-ikz}\right) = \frac{k}{\gamma^2}\vec{\nabla}_\perp\psi\cos kz = \frac{p\pi}{d}\frac{1}{\gamma^2}\vec{\nabla}_\perp\psi\cos\frac{p\pi z}{d}$$

so $H_z(z=0,d) = 0$ as required by $\hat{n}\cdot\vec{H}|_S = 0$.

the eigenvalue equation $\gamma^2 = \mu\varepsilon\omega^2 - k^2 = \mu\varepsilon\omega^2 - (p\pi/d)^2$

and the eigenvalues γ_λ^2 depend on the cylinder boundaries.

For each value of p and eigenvalue γ_λ^2 there is an eigenfrequency

$$\omega_{\lambda p} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\gamma_\lambda^2 + \left(\frac{p\pi}{d}\right)^2}$$

e.g. cylinder of circular cross-section, radius R

TM modes : $\psi(\rho=R)=0$

using cylindrical coordinates $\left(\vec{\nabla}_\perp^2+\gamma^2\right)\psi=0 \to \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+\gamma^2\right)\psi=0$

propose $\psi(\rho,\phi)=Af_m(\rho)\,e^{\pm im\phi}$ then $\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)f_m-\left(m^2+\gamma^2\rho^2\right)f_m=0$

$$\Rightarrow \rho^2 f_m''+\rho f_m'+\left(\gamma^2\rho^2-m^2\right)f_m=0$$

Bessel's eqn

$$\psi(\rho,\phi)=A\,J_m(\gamma\rho)\,e^{\pm im\phi}$$

$\xrightarrow[\text{boundary condition}]{}$ $0=J_m(\gamma R) \Rightarrow \gamma R=x_{mn}$ = the n^{th} root of J_m

$$\underline{\gamma_{mn}=\frac{x_{mn}}{R}.} \qquad \omega_{mnp}=\frac{1}{\sqrt{\mu\epsilon}}\sqrt{\frac{x_{mn}^2}{R^2}+\frac{p^2\pi^2}{d^2}}$$

TE modes : $\left.\frac{\partial\psi}{\partial\rho}\right|_{\rho=R}=0 \Rightarrow 0=J_m'(\gamma R) \Rightarrow \gamma R=x_{mn}'$ = the n^{th} root of J_m'

power losses & Q

solving for the eigenfrequencies assuming perfectly conducting boundaries was an approximation - in this approximation energy can only be supplied to a cavity if the system is driven at EXACTLY one of these frequencies, and once the energy is in there, it will never dissipate.

In practice of course, the walls are not perfectly conducting & there is energy loss by Joule heating.

We can characterise the energy loss using the "Q" variable,

$$Q = 2\pi \frac{\langle \text{stored energy} \rangle}{\text{energy loss per cycle}} = \omega_0 \frac{\langle U \rangle}{P_{loss}}$$

P_{loss} ← power loss

since $P_{loss} = -\frac{dU}{dt}$ we have $U(t)$ satisfying $\frac{dU}{dt} = -\frac{\omega_0}{Q} U$

so if we start with energy U_0 stored, over time it will dissipate exponentially

$$U(t) = U_0 e^{-\frac{\omega_0}{Q} t}$$

In general the fields in the cavity will have time dependence like

$$E(t) = E_0 e^{-\frac{\omega_0}{2Q} t} e^{-i(\omega_0 + \Delta\omega)t}$$

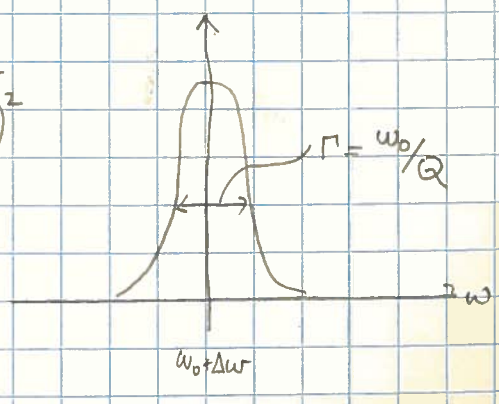
where a frequency shift is allowed for

the corresponding frequency distribution is

$$E(\omega) = \int_0^\infty dt E(t) e^{i\omega t} = E_0 \int_0^\infty dt \exp\left[t\left(i(\omega - \omega_0 - \Delta\omega) - \frac{\omega_0}{2Q} \right) \right]$$

from which we determine

$$|E(\omega)|^2 \propto \frac{1}{(\omega - \omega_0 + \Delta\omega)^2 + (\omega_0/2Q)^2}$$



→ Q for the cylindrical cavity

TM modes : $E_z = \psi(x,y)\cos\frac{p\pi z}{d}$ / $\vec{E}_\perp = -\frac{p\pi}{d\gamma^2}\sin\frac{p\pi z}{d}\vec{\nabla}_\perp\psi$ / $\vec{H}_\perp = i\frac{\epsilon\omega}{\gamma^2}\cos\frac{p\pi z}{d}\hat{z}\times\vec{\nabla}_\perp\psi$

the stored energy is $U = \frac{1}{4}\int_V d^3\vec{r}\left(\epsilon|\vec{E}|^2 + \mu|\vec{H}|^2\right)$

in computing this we'll encounter integrals $\int_0^d dz\cos^2\left(\frac{p\pi z}{d}\right) = \frac{1}{2}d\eta_p$ $\quad \eta_p = \begin{cases}2 & p=0\\ 1 & p\neq 0\end{cases}$

$\int_0^d dz\sin^2\left(\frac{p\pi z}{d}\right) = \frac{1}{2}d$

$\underbrace{\int}_{\text{cross section}} dA\,|\vec{\nabla}_\perp\psi|^2$

the 2dim version of Green's 1st identity reads

$$\int dA\left(\phi\nabla_\perp^2\psi + \vec{\nabla}_\perp\psi\cdot\vec{\nabla}_\perp\phi\right) = \oint_C dl\,\phi\frac{\partial\psi}{\partial n}$$

so choosing $\phi = \psi^*$ we obtain

$$\int dA\,|\vec{\nabla}_\perp\psi|^2 = \underbrace{\oint_C dl\,\psi^*\frac{\partial\psi}{\partial n}}_{\text{zero by b.c}} - \underbrace{\int dA\,\psi^*\nabla_\perp^2\psi}_{-\gamma^2\int dA|\psi|^2 \text{ by eigenequation}}$$

$$\int dA\,|\vec{\nabla}_\perp\psi|^2 = \gamma^2\int dA\,|\psi|^2$$

we find $U = \frac{1}{4}\epsilon\int dA|\psi|^2\cdot\frac{1}{2}d\eta_p$

$+\frac{1}{4}\epsilon\int dA|\psi|^2\cdot\gamma^2\cdot\frac{1}{2}d\cdot\left(\frac{p\pi}{d}\right)^2\frac{1}{\gamma^4}$

$+\frac{1}{4}\mu\int dA|\psi|^2\cdot\gamma^2\cdot\frac{1}{2}d\eta_p\cdot\frac{\epsilon^2\omega^2}{\gamma^4}$ $\quad \mu\epsilon\omega^2 = \gamma^2 + k^2 = \gamma^2 + (p\pi/d)^2$

$$U = \frac{1}{4}\epsilon\int dA|\psi|^2\,\frac{1}{2}d\left(\eta_p + \left(\frac{p\pi}{d\gamma}\right)^2 + \eta_p\left(1 + \left(\frac{p\pi}{d\gamma}\right)^2\right)\right)$$

$$\boxed{U = \frac{d}{4}\epsilon\int dA|\psi|^2\left[\eta_p + \left(\frac{p\pi}{d\gamma}\right)^2\right]}$$

the power loss can be found using the integrated loss through the effective surface current:

$$\frac{dP}{dA} = \frac{1}{2}\frac{1}{\sigma\delta}\left|\hat{n}\times\vec{H}\right|^2$$

$$P = \frac{1}{2\sigma\delta}\oint dl \int_0^d dz \left|\hat{n}\times\vec{H}\right|^2_{cyl} + \frac{1}{2\sigma\delta}2\int dA \left|\hat{n}\times\vec{H}\right|^2_{end}$$

$$P = \frac{\epsilon}{\sigma\delta\mu}\left[1+\left(\frac{p\pi}{\gamma d}\right)^2\right]\int dA\,|\psi|^2\cdot\left[1+\xi\frac{Cd}{4A}\right]$$

where ξ is a dimensionless number, C is the circumference and A the area [see Jackson for the explicit calculation]

and thus (for $p\neq0$)

$$Q = \omega\frac{U}{P_{loss}} = \omega\cdot\frac{d\epsilon/4\left(1+\left(\frac{p\pi}{\gamma d}\right)^2\right)\int dA\,|\psi|^2}{\epsilon/\sigma\delta\mu\left(1+\left(\frac{p\pi}{\gamma d}\right)\right)\int dA\,|\psi|^2}\cdot\frac{1}{1+\xi\frac{Cd}{4A}}$$

$$\delta = \sqrt{\frac{2}{\mu_c\sigma\omega}}$$

$$\sigma = \frac{2}{\mu_c\omega\delta^2}$$

$$= \frac{d.\omega}{4}\cdot\frac{2\delta\mu}{\mu_c\omega\delta^2}\cdot\frac{1}{1+\xi\,Cd/4A}$$

$$= \frac{\mu}{\mu_c}\frac{d}{\delta}\cdot\frac{1}{2}\frac{1}{1+\xi\,Cd/4A}$$

$$Q = \frac{\mu}{\mu_c}\cdot\left(\frac{V}{S\delta}\right)\cdot\left(\begin{matrix}\text{geometric}\\ \text{factor}\end{matrix}\right)$$

ie ratio of volume occupied by fields to the volume of conductor into which fields penetrate