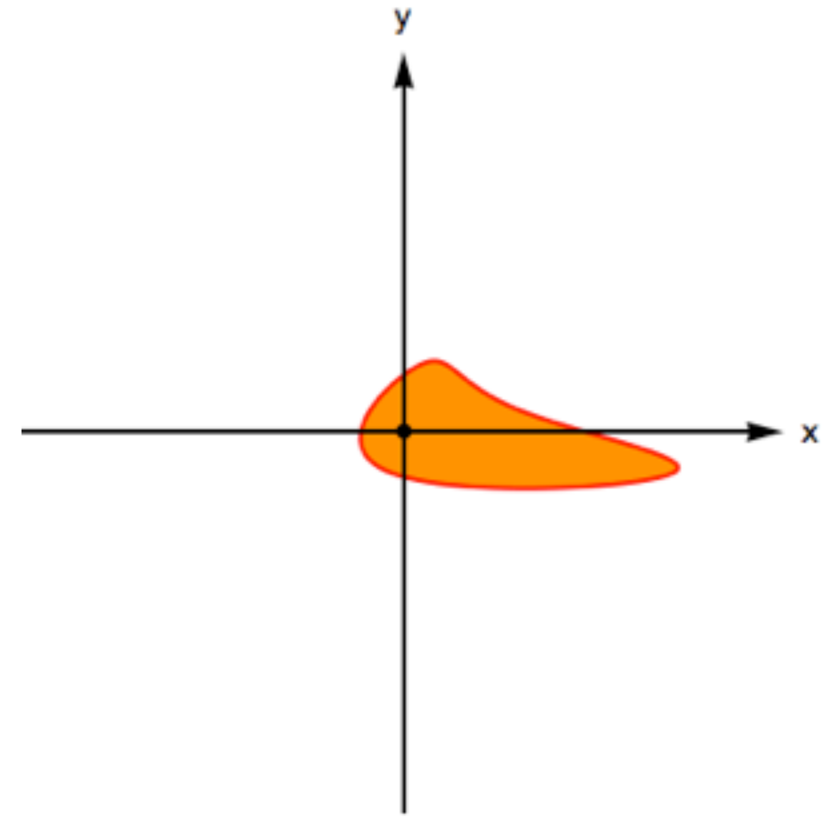
rotational motion

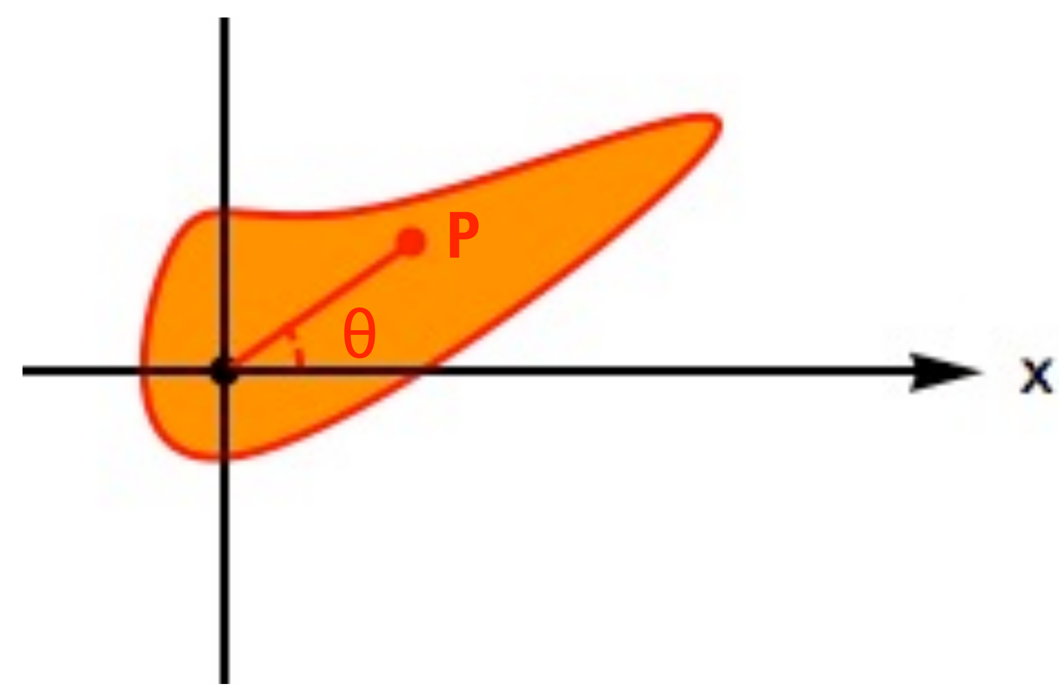
rotations of a rigid body

➜ suppose we have a body which rotates about some axis



➜ we can define its orientation at any moment by an angle, θ

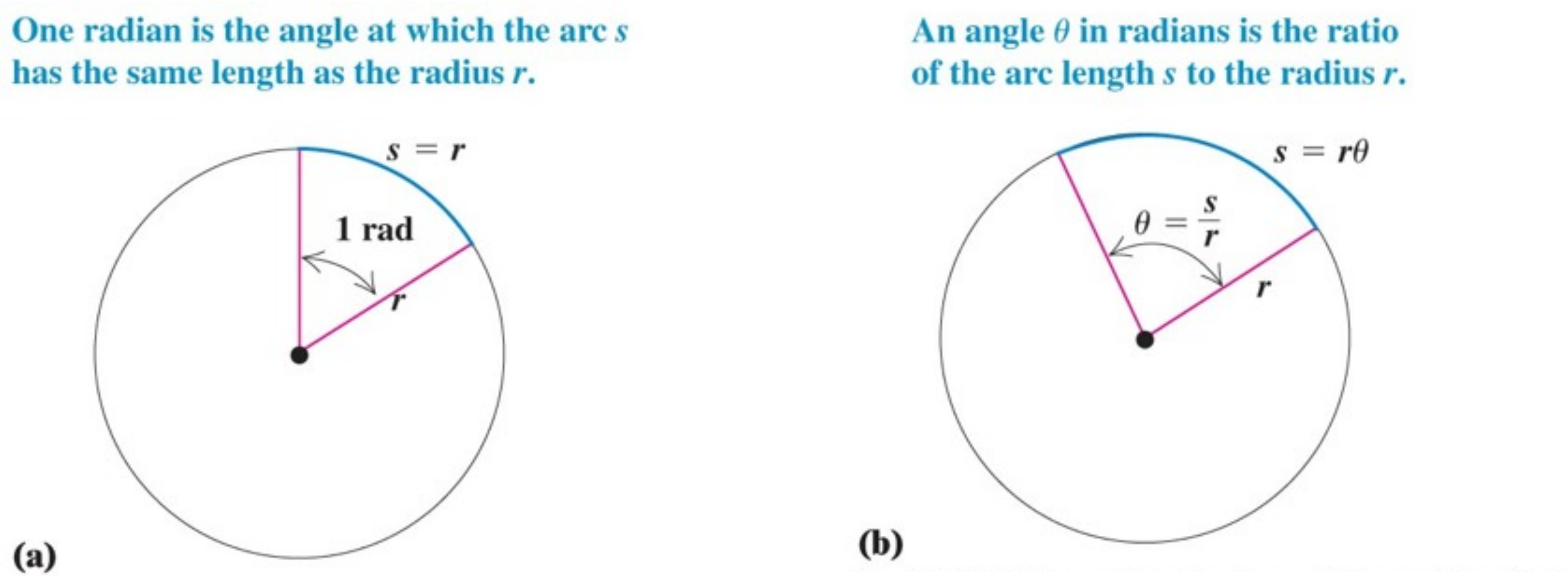
(any point P will do)



radians

➜ measuring θ in degrees turns out to be a poor choice

➜ **radians** are a more natural choice of angular unit



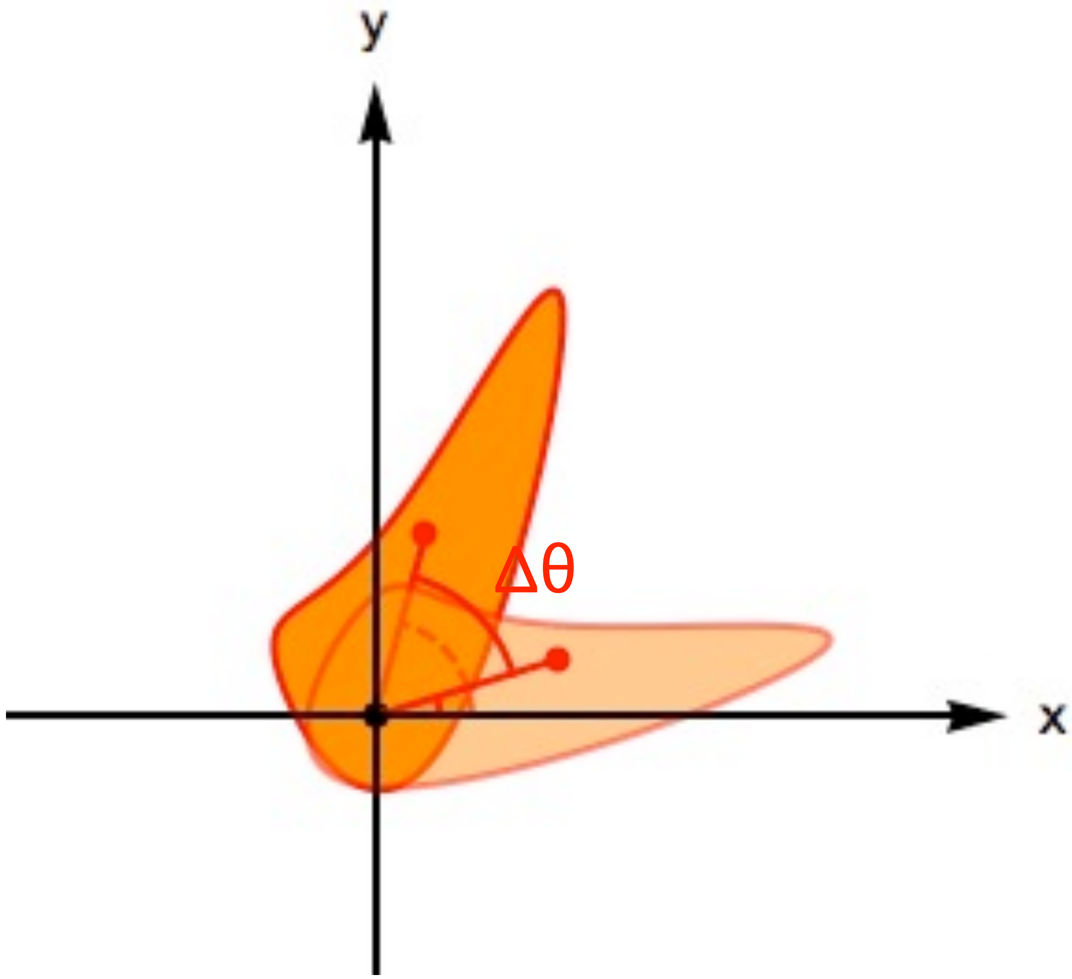
$$1\,\mathrm{rad} = \frac{360^\circ}{2\pi} = 57.3^\circ$$

angular velocity

➜ describe the rate of rotation by the change in angle in a given time

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{\theta_2 - \theta_1}{t_2 - t_1}$$

(notice, just like linear motion but with x→θ)



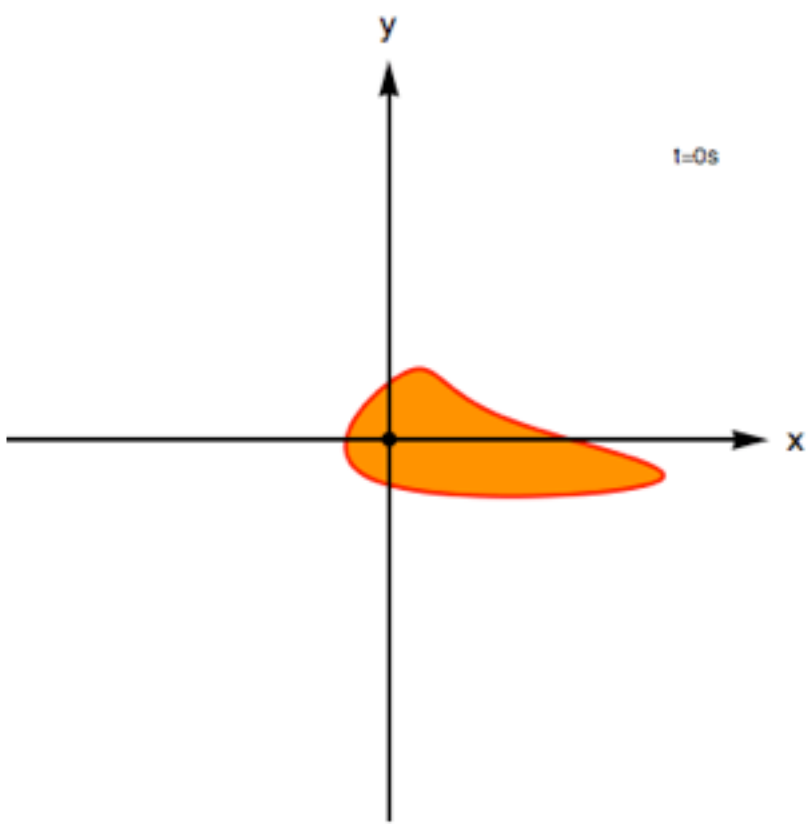
angular velocity

➜ describe the rate of rotation by the change in angle in a given time

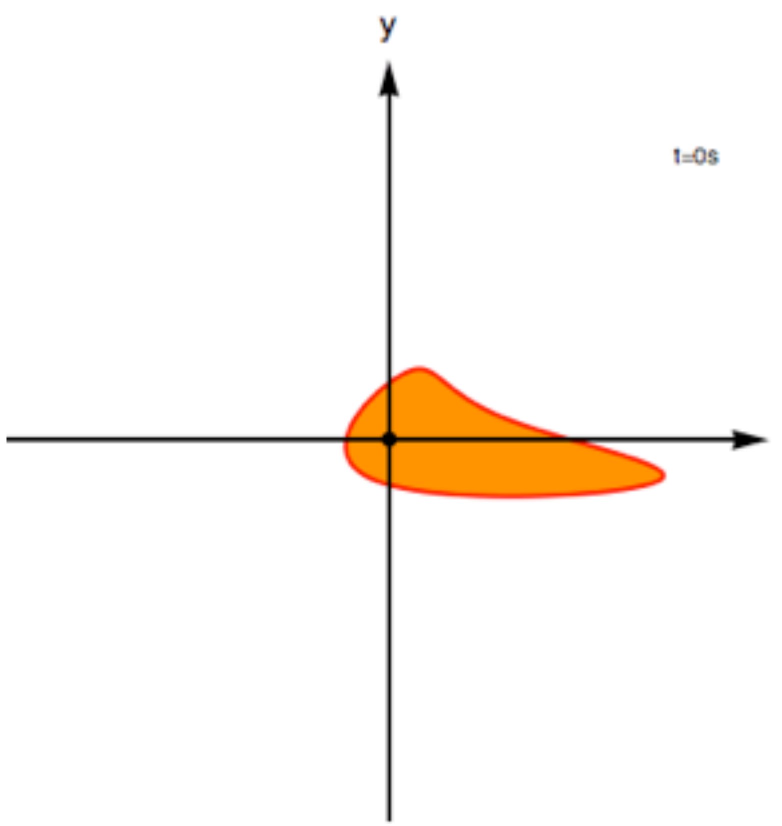
$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{\theta_2 - \theta_1}{t_2 - t_1}$$

(notice, just like linear motion but with $x \rightarrow \theta$)

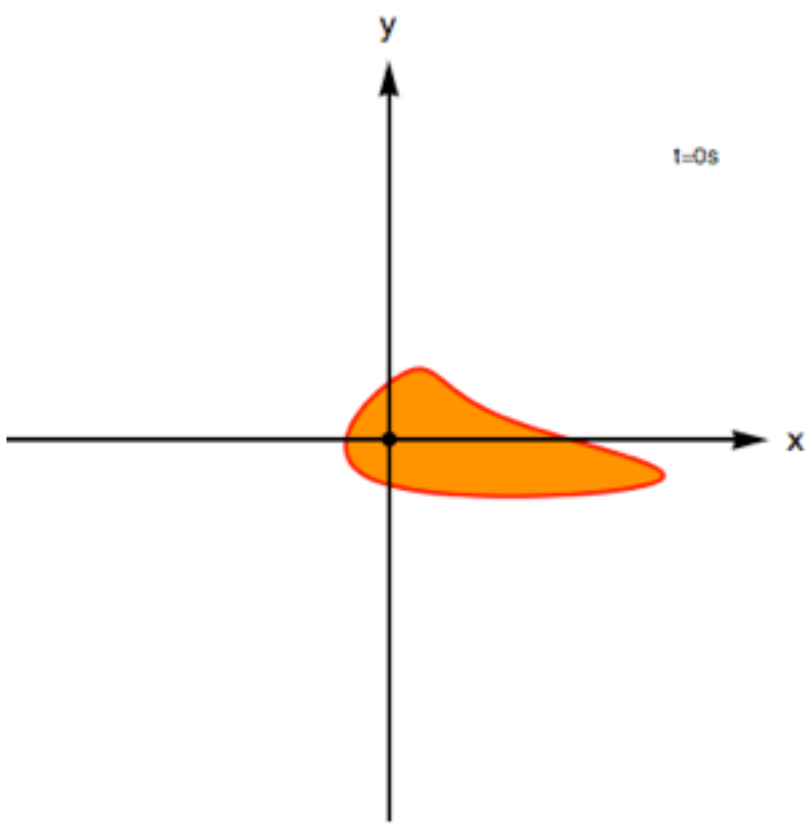
small positive constant ω



larger positive constant ω



negative constant ω



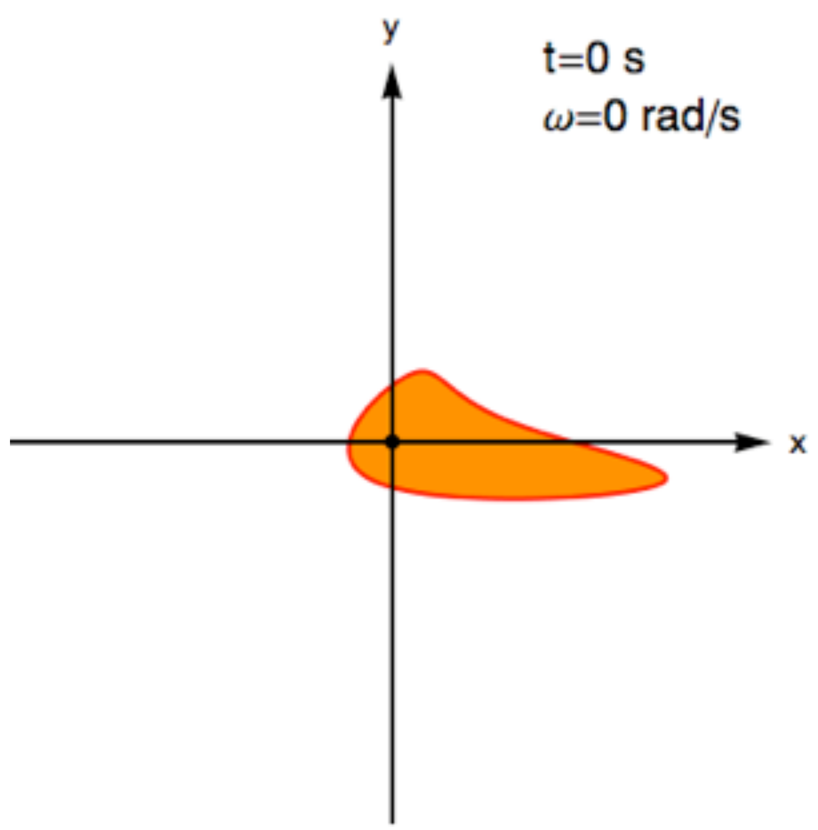
angular acceleration

➜ suppose the rate of rotation changes - we need angular acceleration

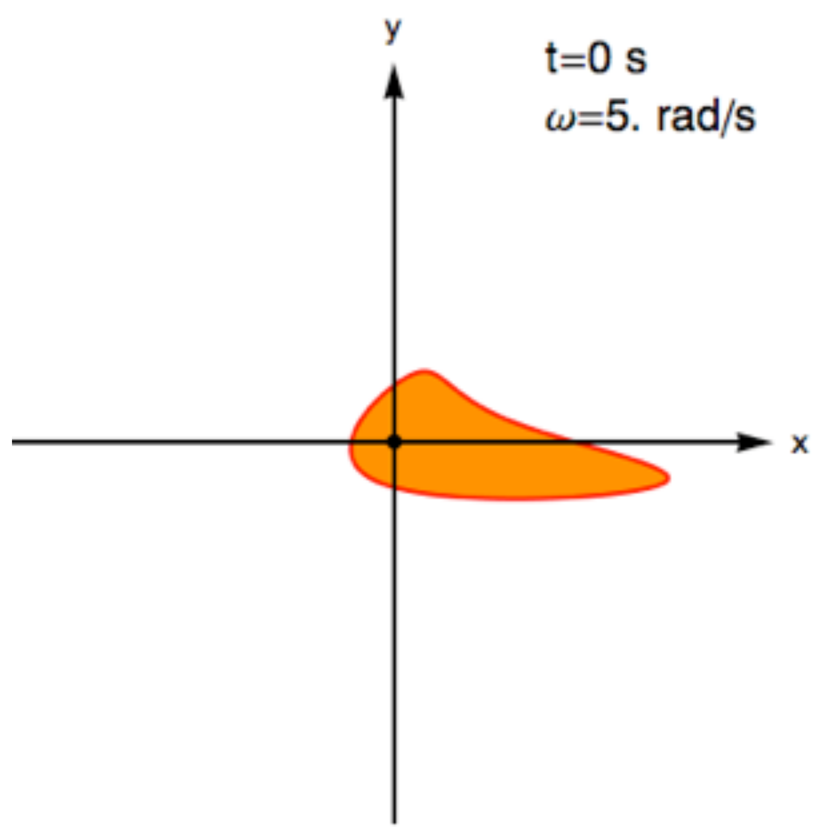
$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{\omega_2 - \omega_1}{t_2 - t_1}$$

(notice, just like linear motion but with v→ω)

positive constant α



negative constant α
begins with positive ω



angular motion vs. linear motion

➜ the analogy between angular motion & linear motion is strong

➜ for constant acceleration we have

$$v = v_0 + at$$

$$x = x_0 + v_0 t + \tfrac{1}{2} a t^2$$

$$v^2 = v_0^2 + 2a(x - x_0)$$

➜ for constant angular acceleration we have

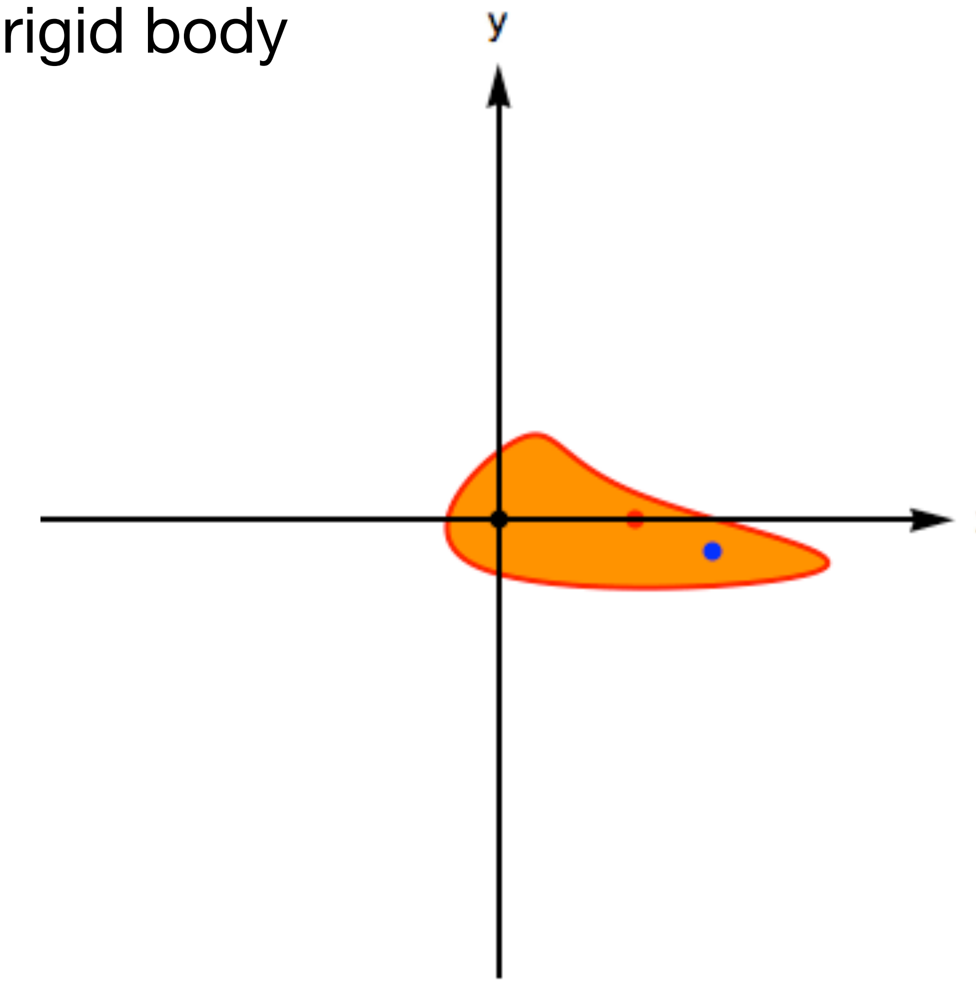
$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2} \alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

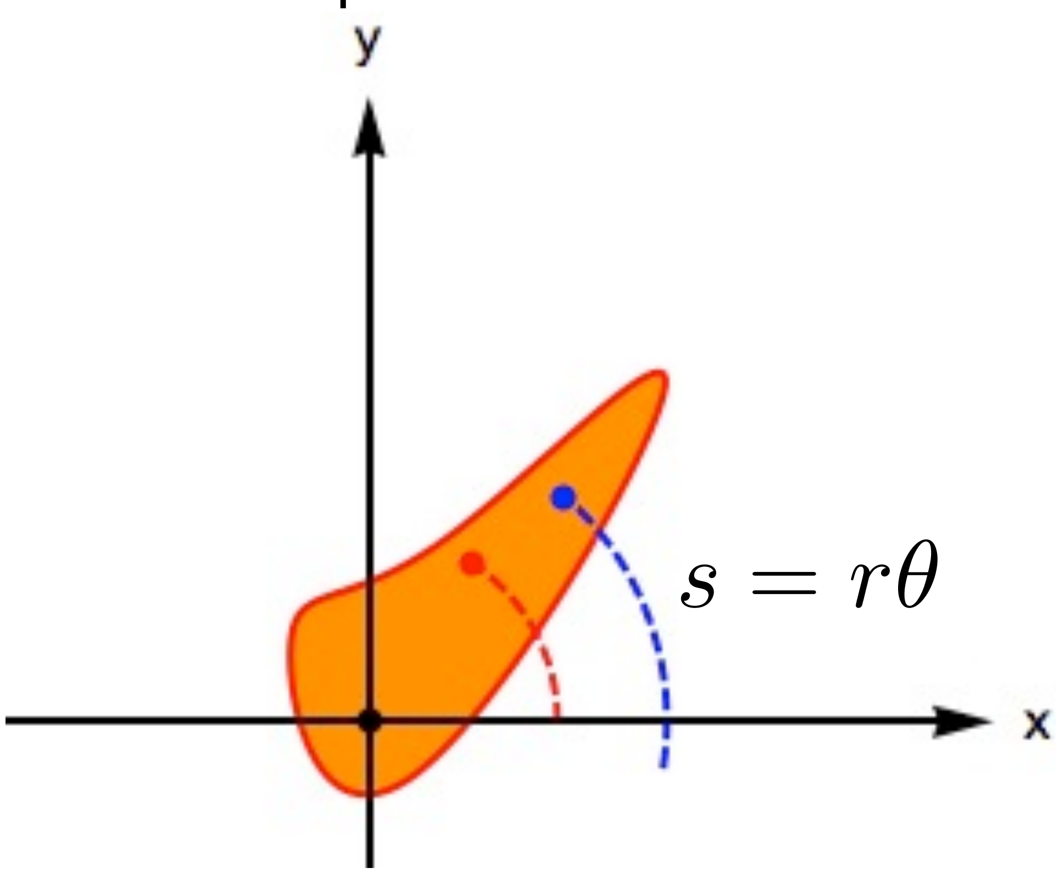
motion of points in a rigid body

➜ consider the motion of a couple of points within the rigid body



the blue point at a large radius travels further in the same time than the red point

so although the angular speed is the same, the linear speed is different

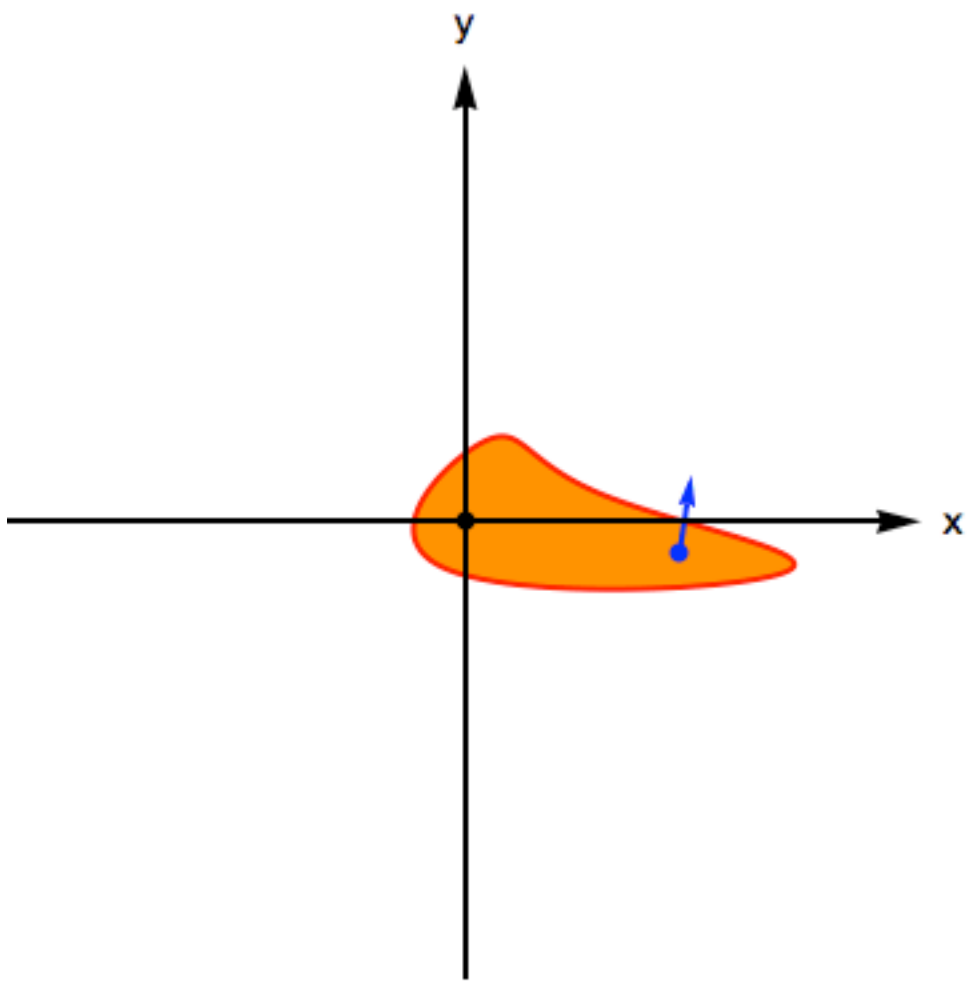


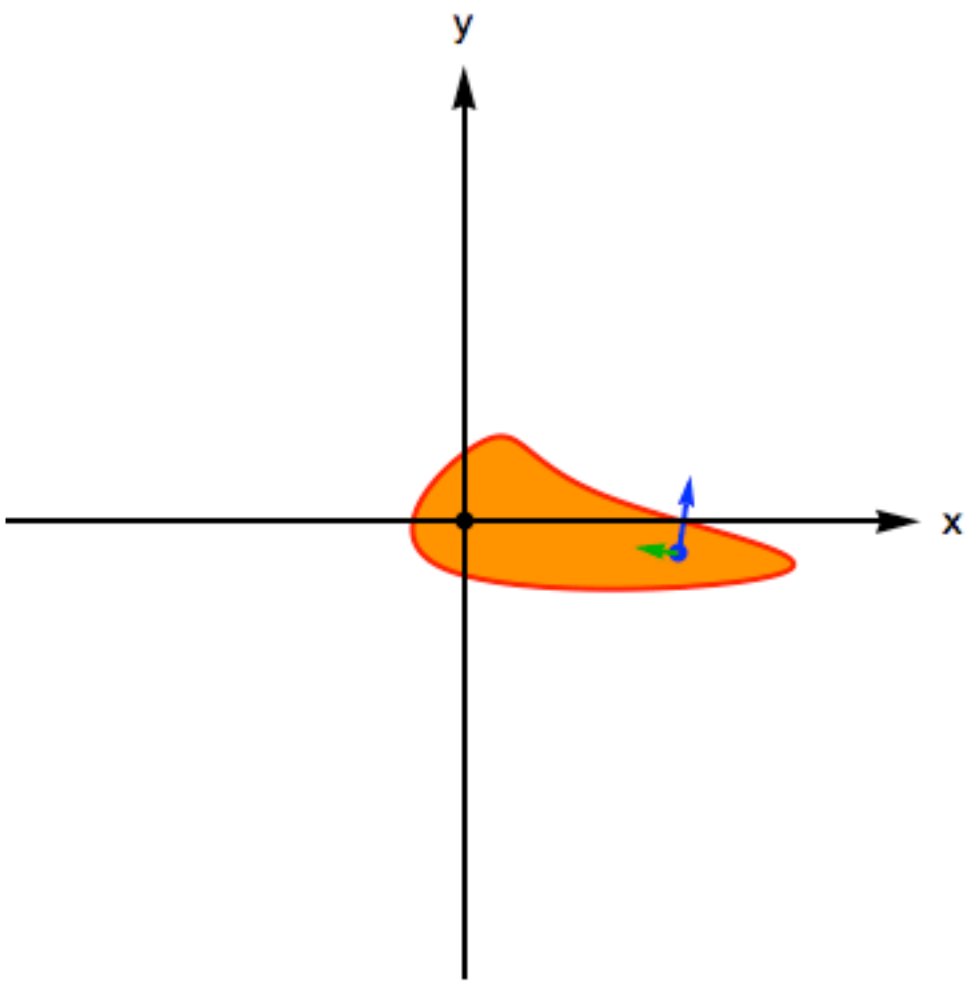
$$s = r\theta$$

$$v = \frac{\Delta s}{\Delta t} = r\frac{\Delta \theta}{\Delta t} = r\omega$$

acceleration of points in a rigid body

➜ consider a rigid body rotating at a constant angular speed (α=0)





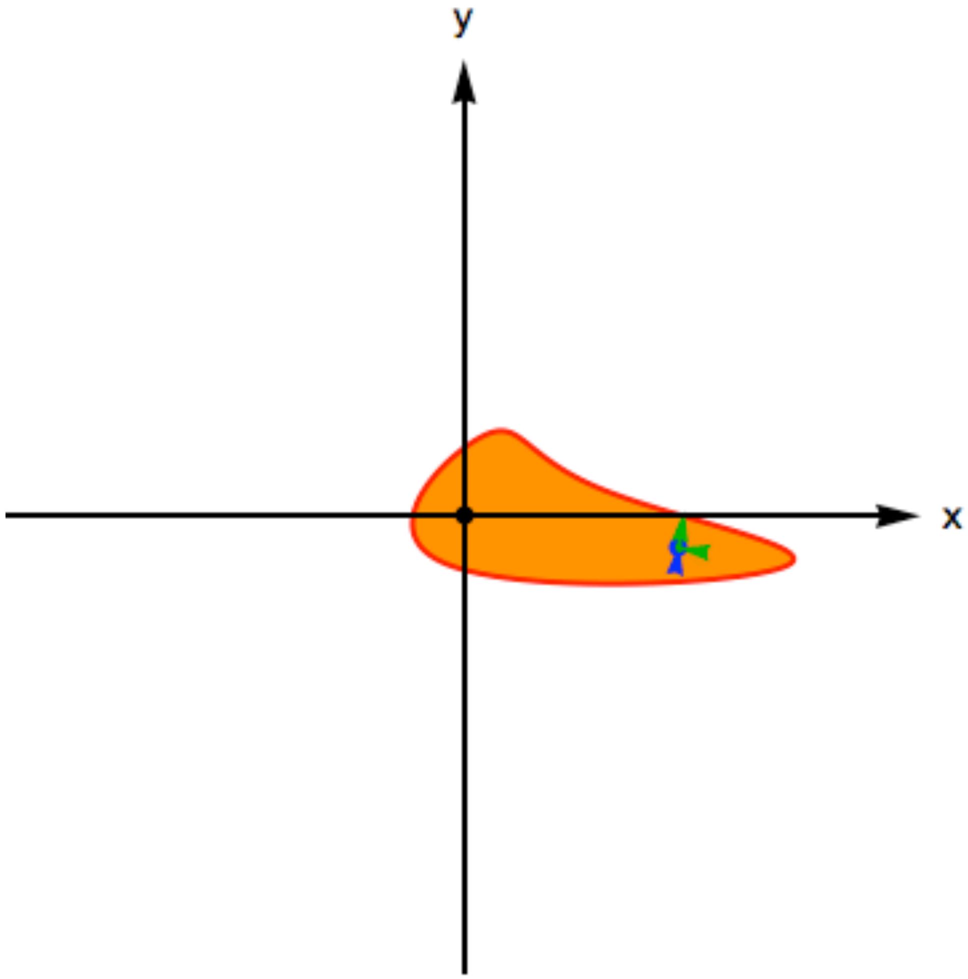
uniform circular motion
- acceleration is radial

acceleration of points in a rigid body

➜ consider a rigid body rotating at a constant angular acceleration ($\alpha \neq 0$)

circular motion
- one component of acceleration is radial
- changing the direction

changing speed
- one component of acceleration is tangential
- changing the speed



acceleration of points in a rigid body

➜ consider a rigid body rotating at a constant angular acceleration (α≠0)

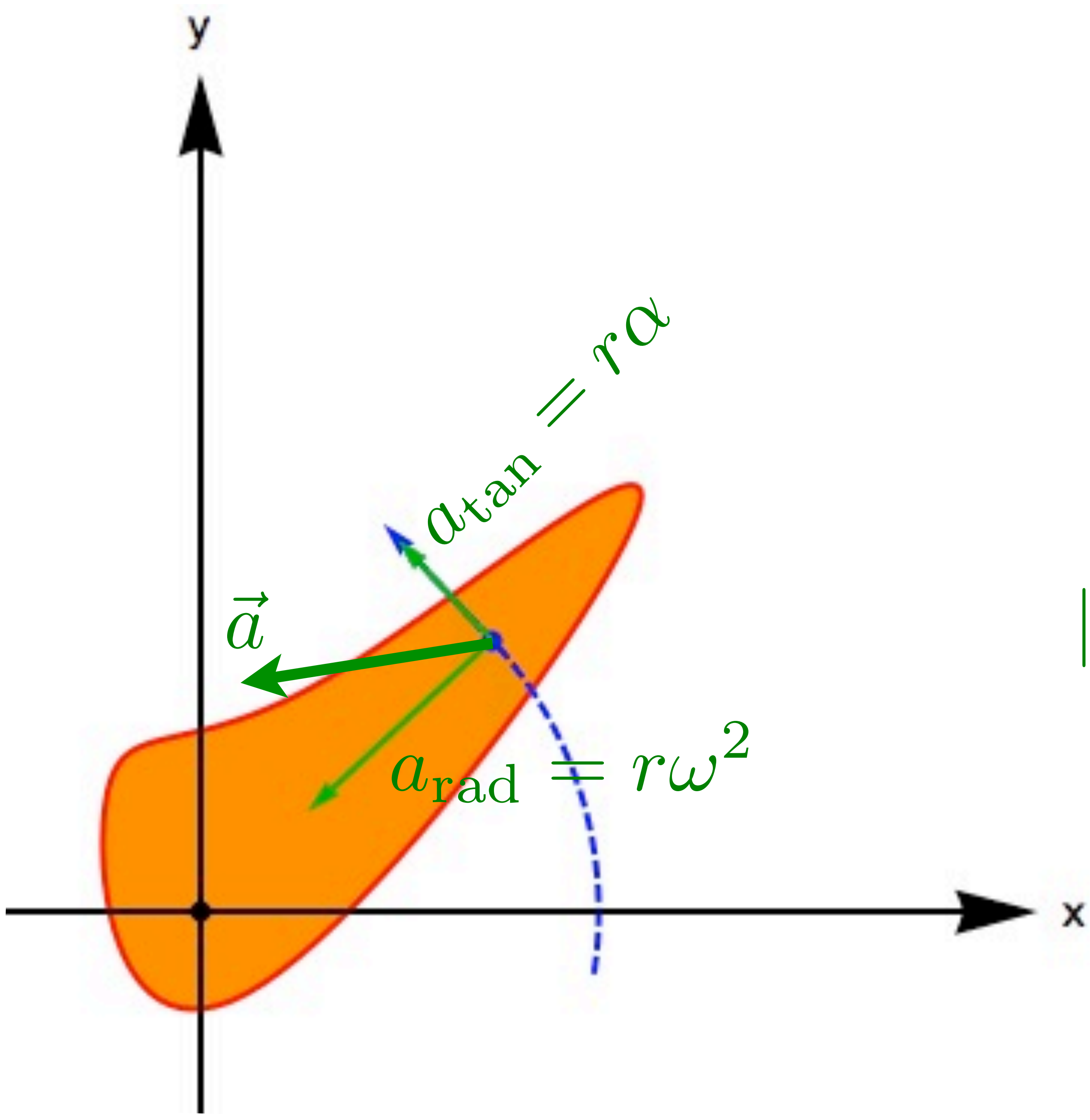
circular motion
- one component of acceleration is radial
- changing the direction

$$a_{\mathrm{rad}} = \frac{v^2}{r} = r\omega^2$$

changing speed
- one component of acceleration is tangential
- changing the speed

$$a_{\mathrm{tan}} = \frac{\Delta v}{\Delta t} = r\frac{\Delta \omega}{\Delta t} = r\alpha$$

➜ acceleration is a combination of radial and tangential components



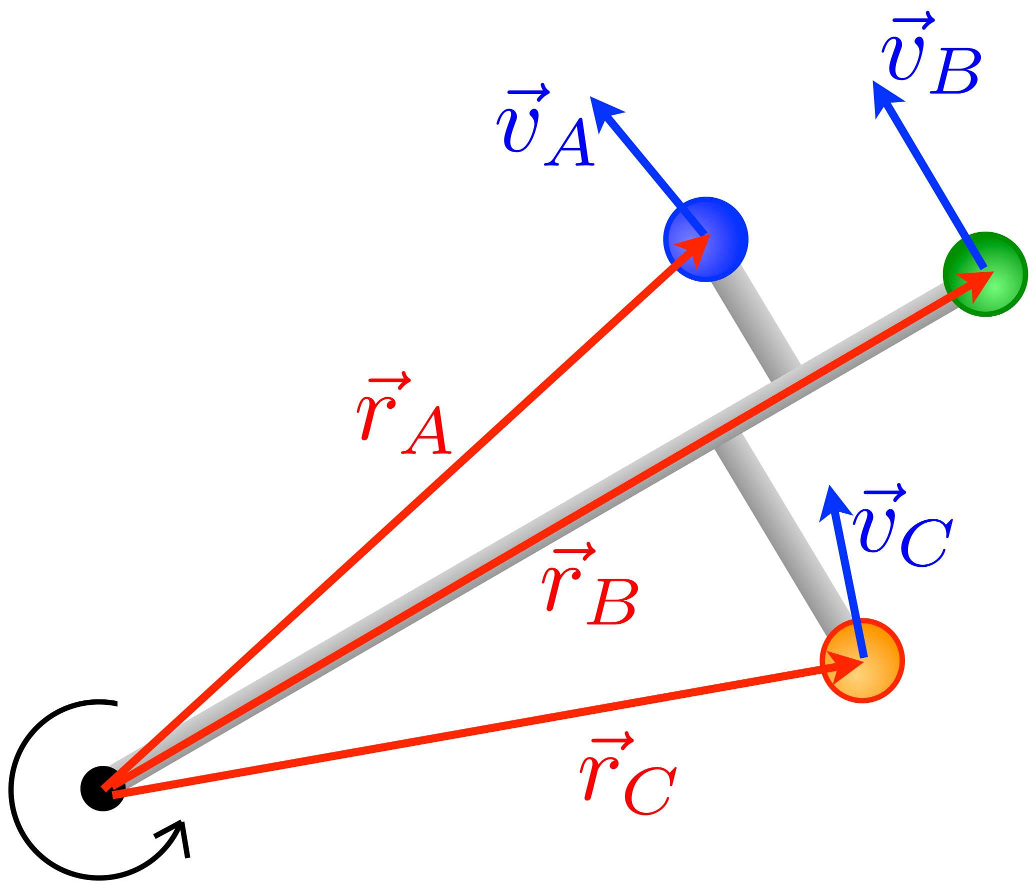
$$|\vec{a}| = \sqrt{a_{\rm rad}^2 + a_{\tan}^2}$$

kinetic energy of rotation

➜ remember that moving objects have kinetic energy

➜ rotating bodies are moving - they must have kinetic energy

➜ consider a rigid body made from massive spheres held together by light rods



rotating in the plane of the page
about this point

$$K = \tfrac{1}{2} m_A v_A^2 + \tfrac{1}{2} m_B v_B^2 + \tfrac{1}{2} m_C v_C^2$$

$$v_A = r_A \omega \qquad \text{etc...}$$

$$K = \tfrac{1}{2} \left(m_A r_A^2 + m_B r_B^2 + m_C r_C^2 \right) \omega^2$$

"moment of inertia"

$$I = m_A r_A^2 + m_B r_B^2 + m_C r_C^2 + \ldots$$

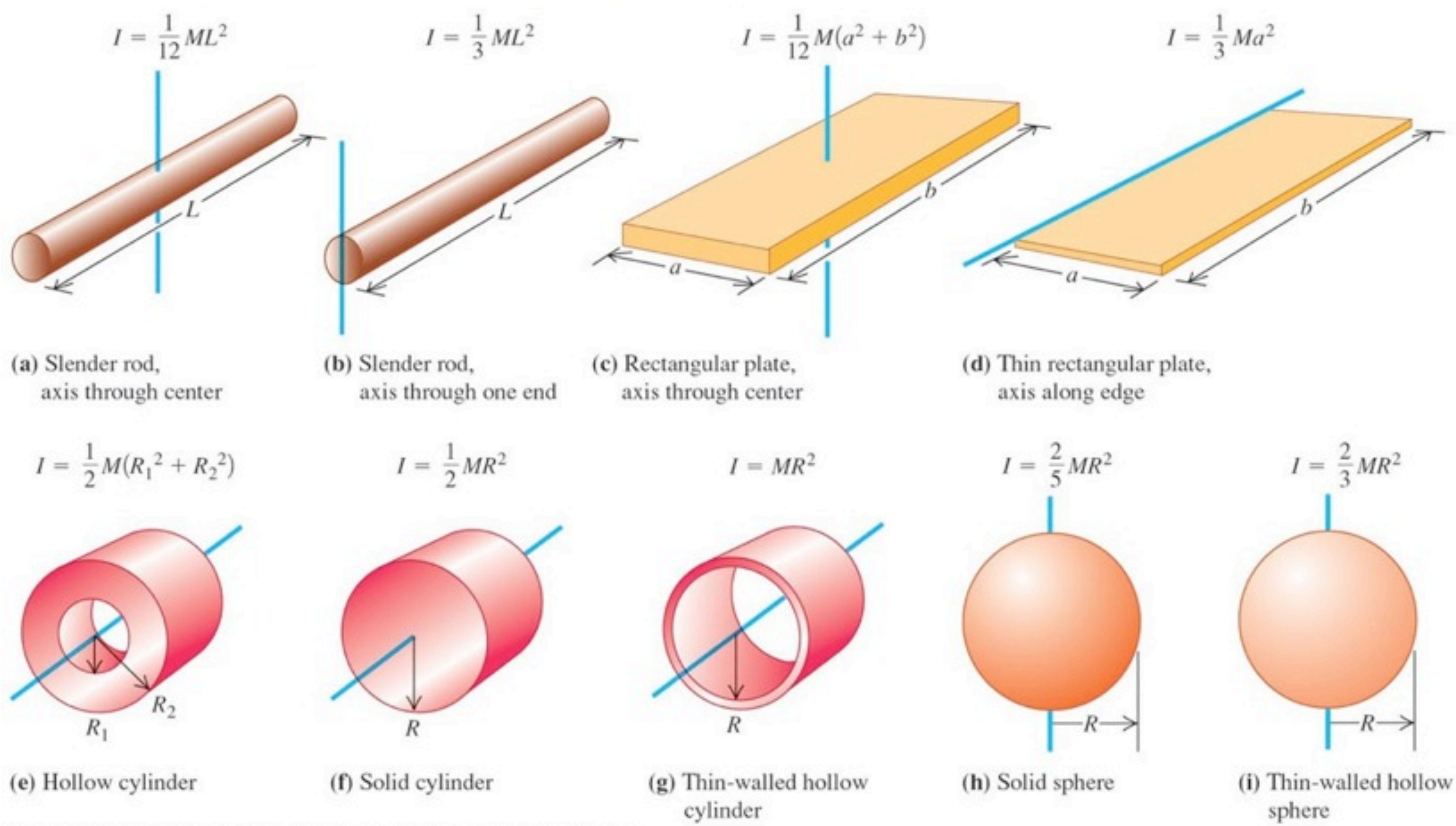
$$K = \tfrac{1}{2} I \omega^2$$

moment of inertia of solid bodies

➜ the moment of inertia of a solid body can be calculated by “adding up” all the particles it is made from (technically an ‘integral’ in calculus)

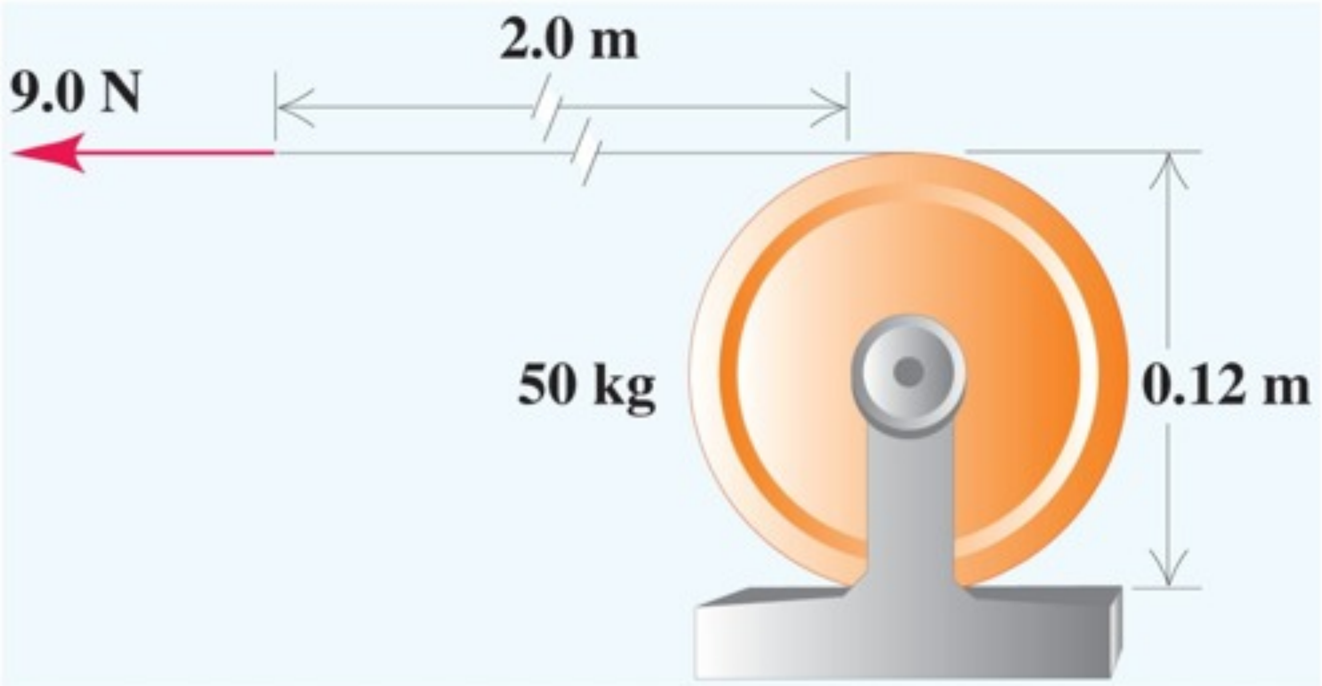
➜ we’ll just use the results of these calculations

TABLE 9.2 Moments of inertia for various bodies



example 9.7

A light, flexible, nonstretching cable is wrapped several times around a winch drum - a solid cylinder of mass 50kg and diameter 0.12m that rotates around a stationary horizontal axis that turns on frictionless bearings. The free end of the cable is pulled with a constant force of magnitude 9.0 N for a distance of 2.0m. It unwinds without slipping, turning the cylinder as it does so. If the cylinder is initially at rest, find its final angular velocity ω and the final speed *v* of the cable

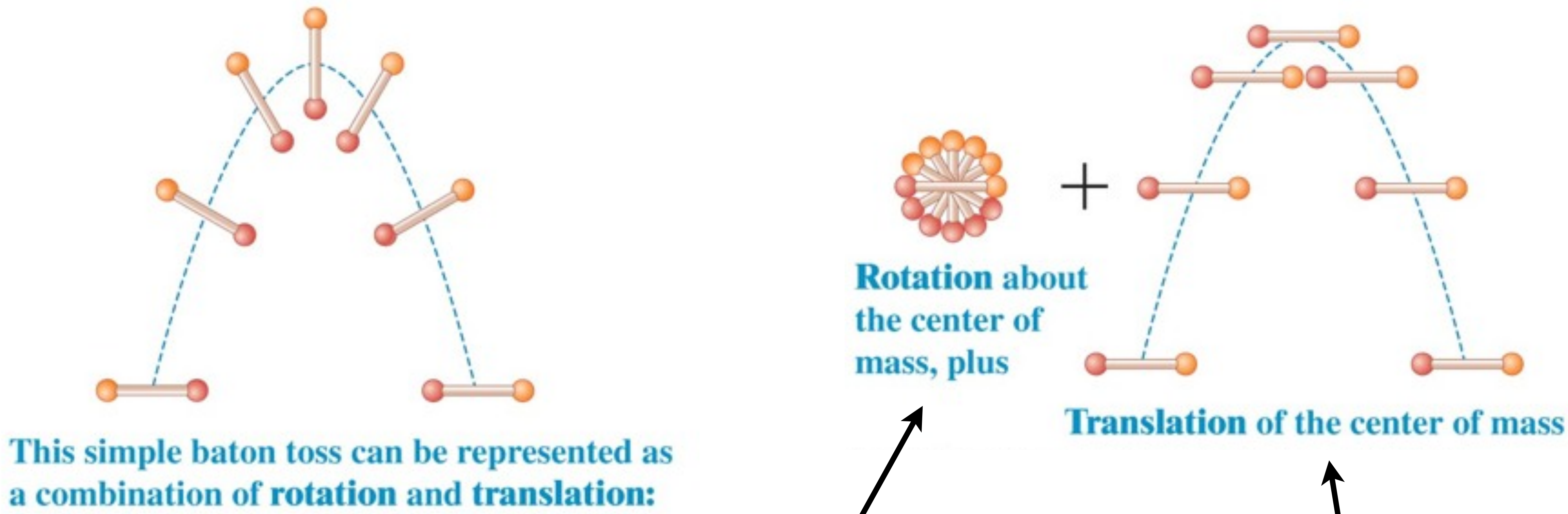


rotation & translation

➜ how can we separate the translation from the rotation ?

center of mass

➜ bodies have a “center-of-mass” which just translates
➜ rotations occur about an axis through the center of mass



$$K_{\text{rot.}} = \tfrac{1}{2} I_{\text{cm}} \omega^2$$

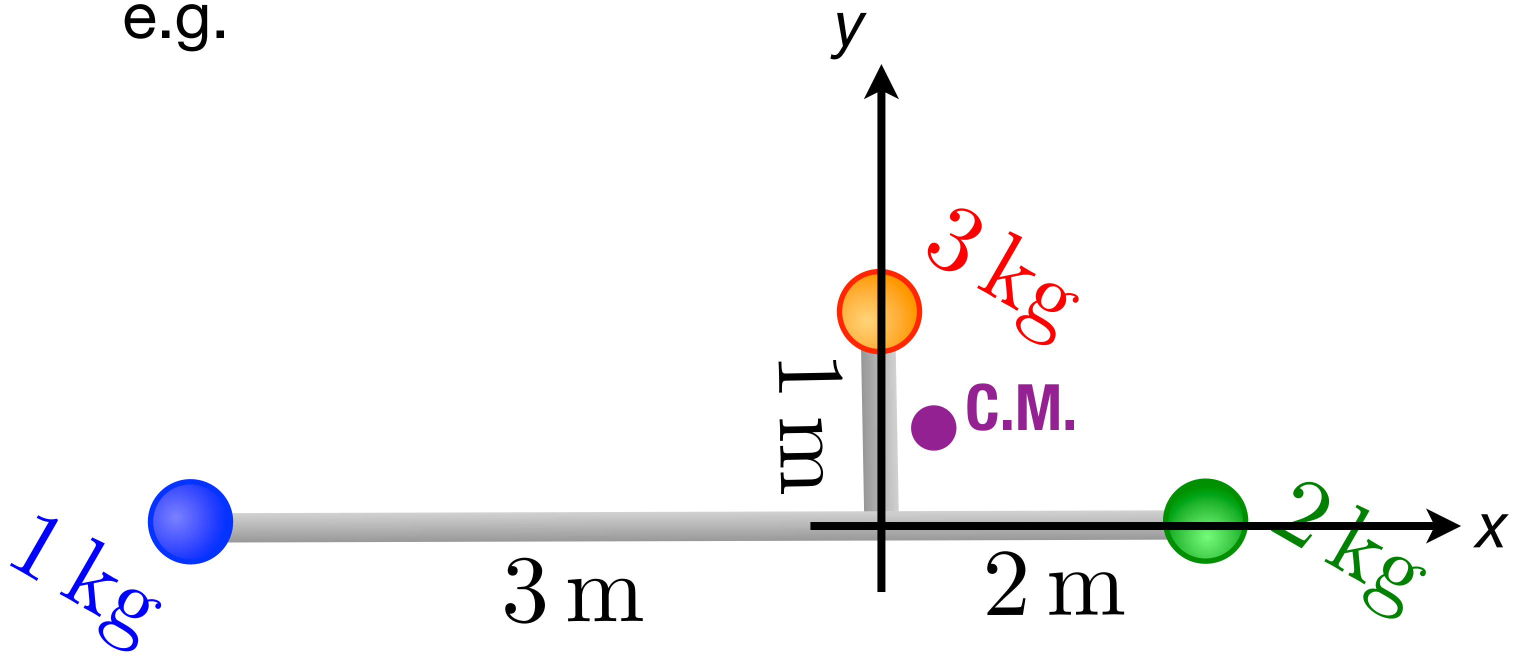
$$K_{\text{cm}} = \tfrac{1}{2} M v_{\text{cm}}^2$$

finding the center of mass

➜ for bodies made of point masses :

$$\vec{R}_{\rm cm} = \frac{m_A\vec{r}_A + m_B\vec{r}_B + m_C\vec{r}_C + \ldots}{m_A + m_B + m_C + \ldots}$$

e.g.



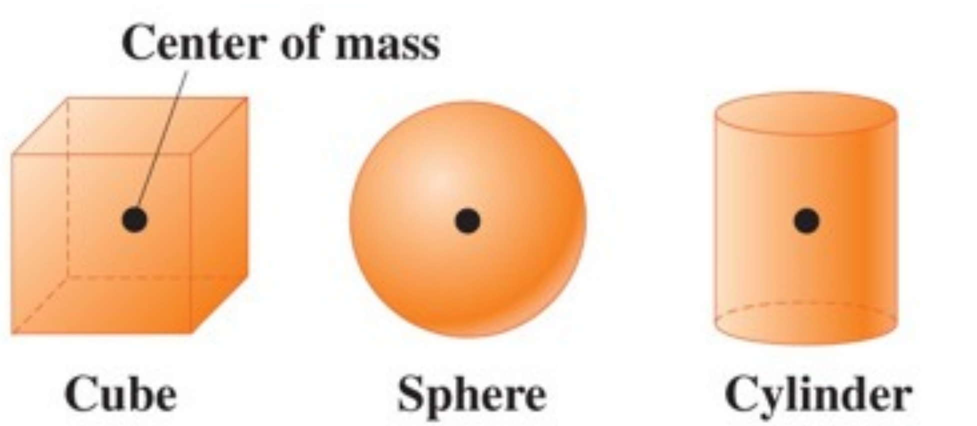
$$X_{\rm cm} = \frac{(1\,{\rm kg})(-3\,{\rm m}) + (2\,{\rm kg})(2\,{\rm m}) + (3\,{\rm kg})(0\,{\rm m})}{6\,{\rm kg}} = \tfrac{1}{6}\,{\rm m}$$

$$Y_{\rm cm} = \frac{(1\,{\rm kg})(0\,{\rm m}) + (2\,{\rm kg})(0\,{\rm m}) + (3\,{\rm kg})(1\,{\rm m})}{6\,{\rm kg}} = \tfrac{1}{2}\,{\rm m}$$

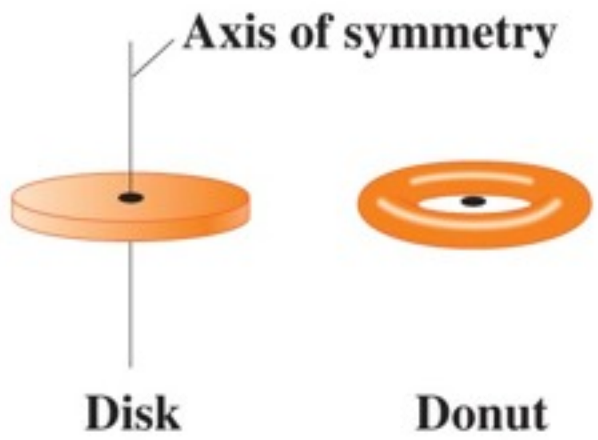
finding the center of mass

➜ for solid bodies need to do calculus

➜ but for uniform solid objects can often guess by symmetry



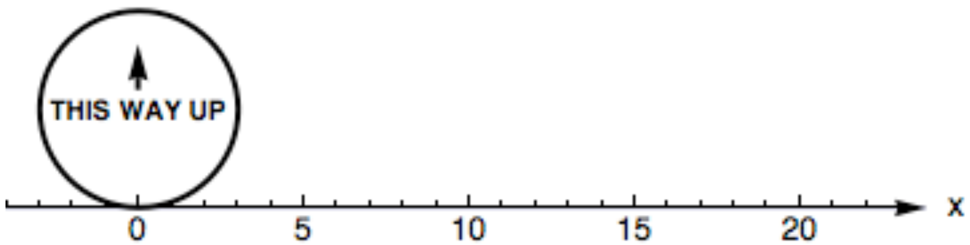
If a homogeneous object has a geometric center, that is where the center of mass is located.

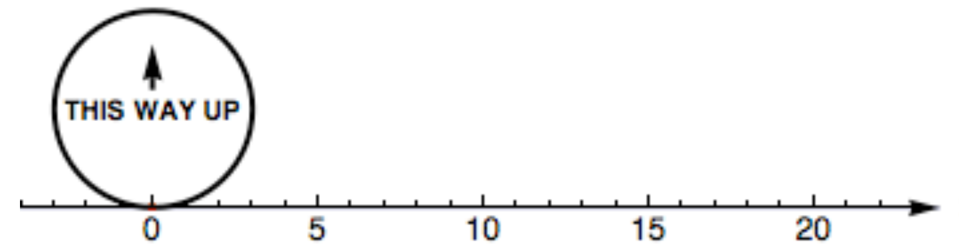


If an object has an axis of symmetry, the center of mass lies along it. As in the case of the donut, the center of mass may not be within the object.

rolling

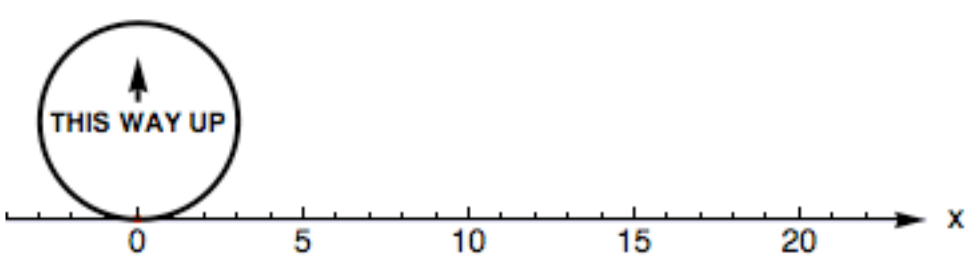
"rolling without slipping"





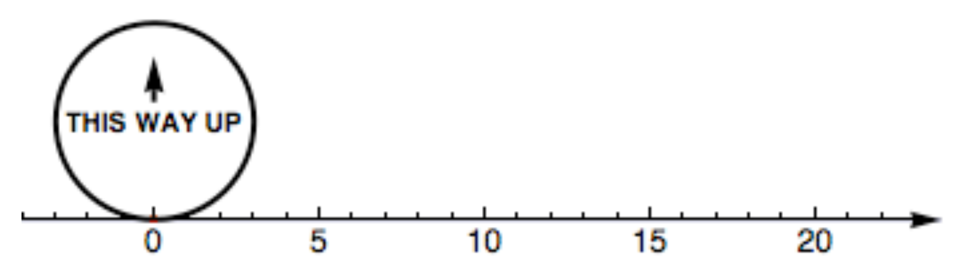
$$v_{\rm cm} = R\omega$$

"wheel spin"



$$v_{\rm cm} < R\omega$$

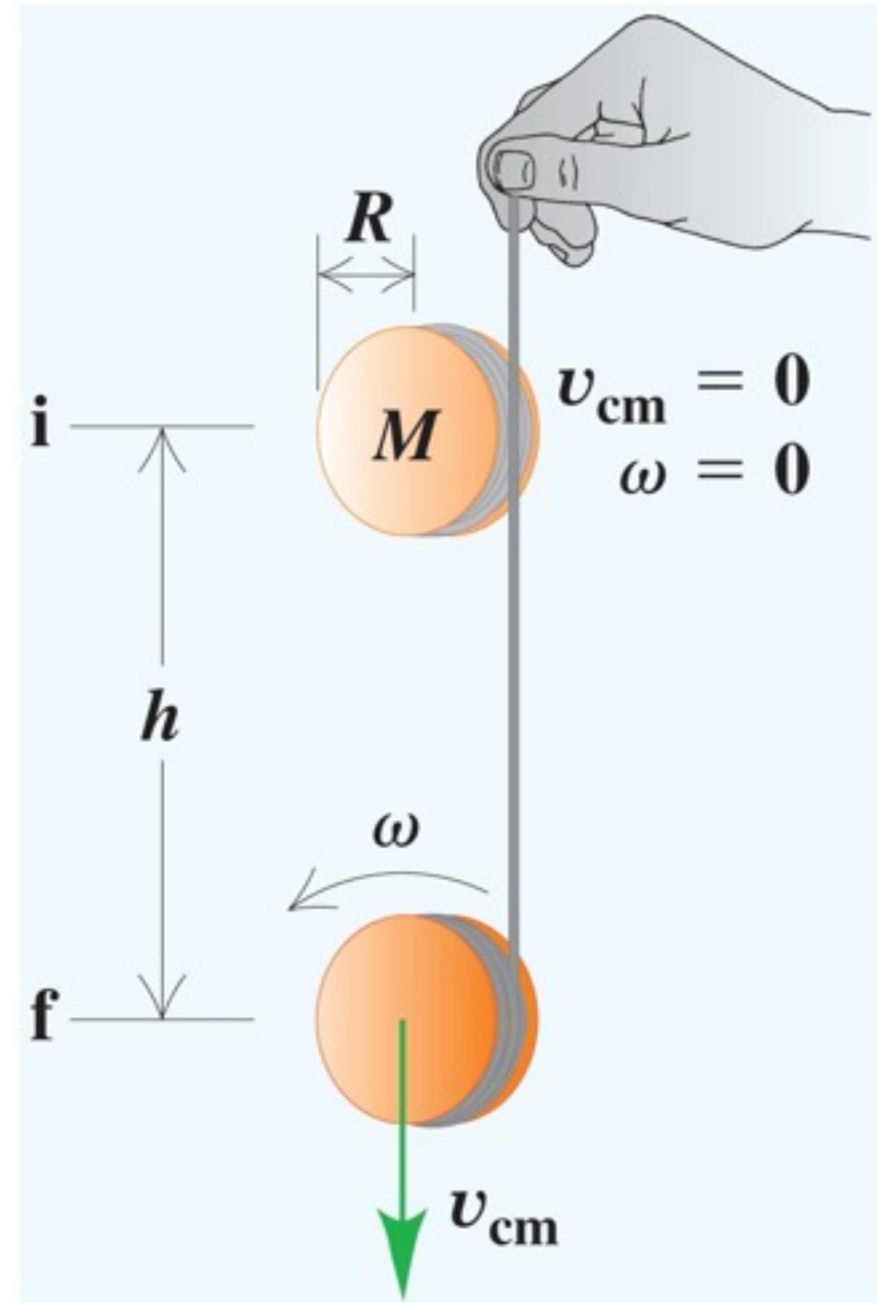
"sliding"



$$v_{\rm cm} > R\omega$$

example 9.9 - a primitive yo-yo

a solid disk of radius *R* and total mass *M* is released from rest with the supporting hand as rest as the string unwinds without slipping. Find an expression for the speed of the center of mass of the disk after it has dropped a distance *h*



kinetic energy $K = \frac{1}{2}Mv_{\rm cm}^2 + \frac{1}{2}I\omega^2$

"without slipping" ⇒ $v_{\rm cm} = R\omega$

solid disk ⇒ $I = \frac{1}{2}MR^2$

$$K = \frac{3}{4}Mv_{\rm cm}^2$$

conservation of energy $K_{\rm i} + U_{\rm i} = K_{\rm f} + U_{\rm f}$

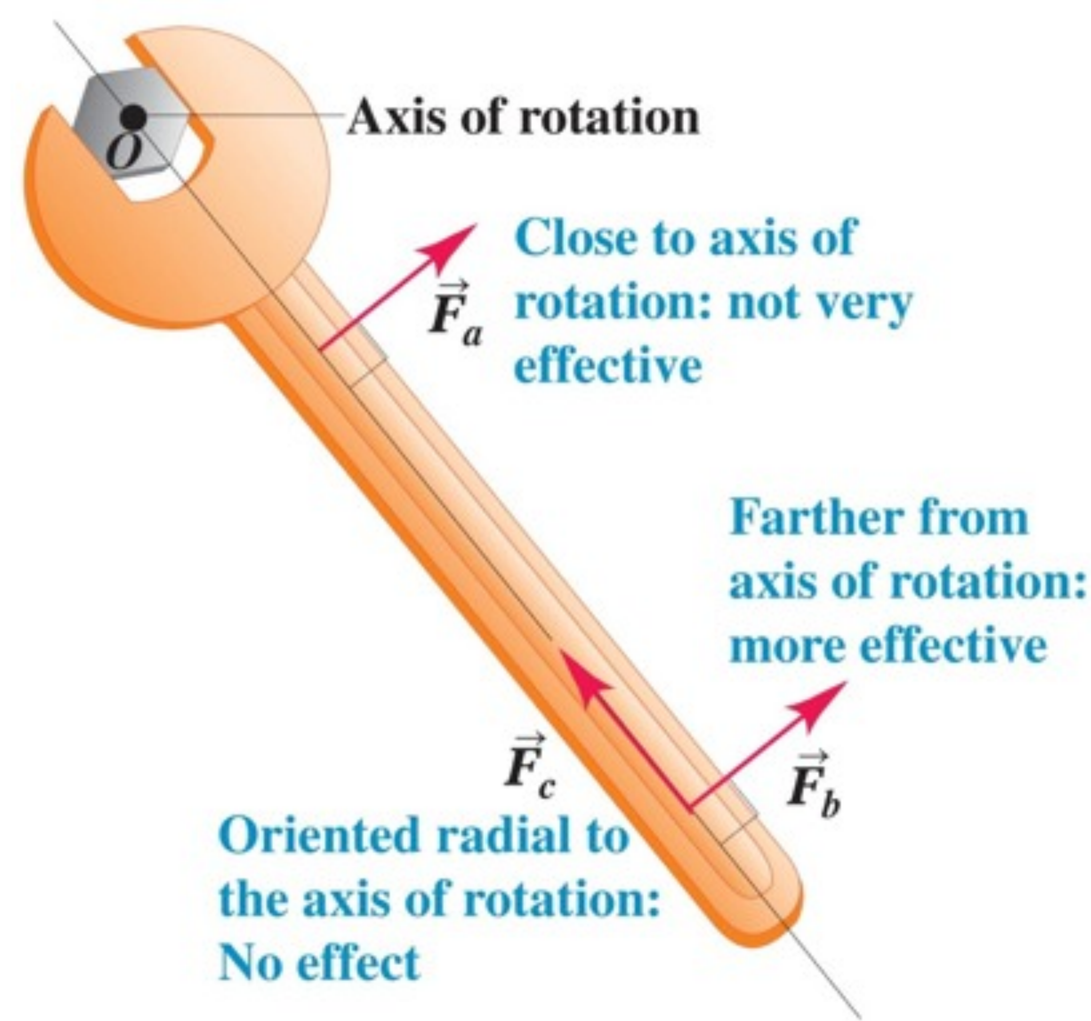
$$0 + Mgh = \frac{3}{4}Mv_{\rm cm}^2 + 0$$

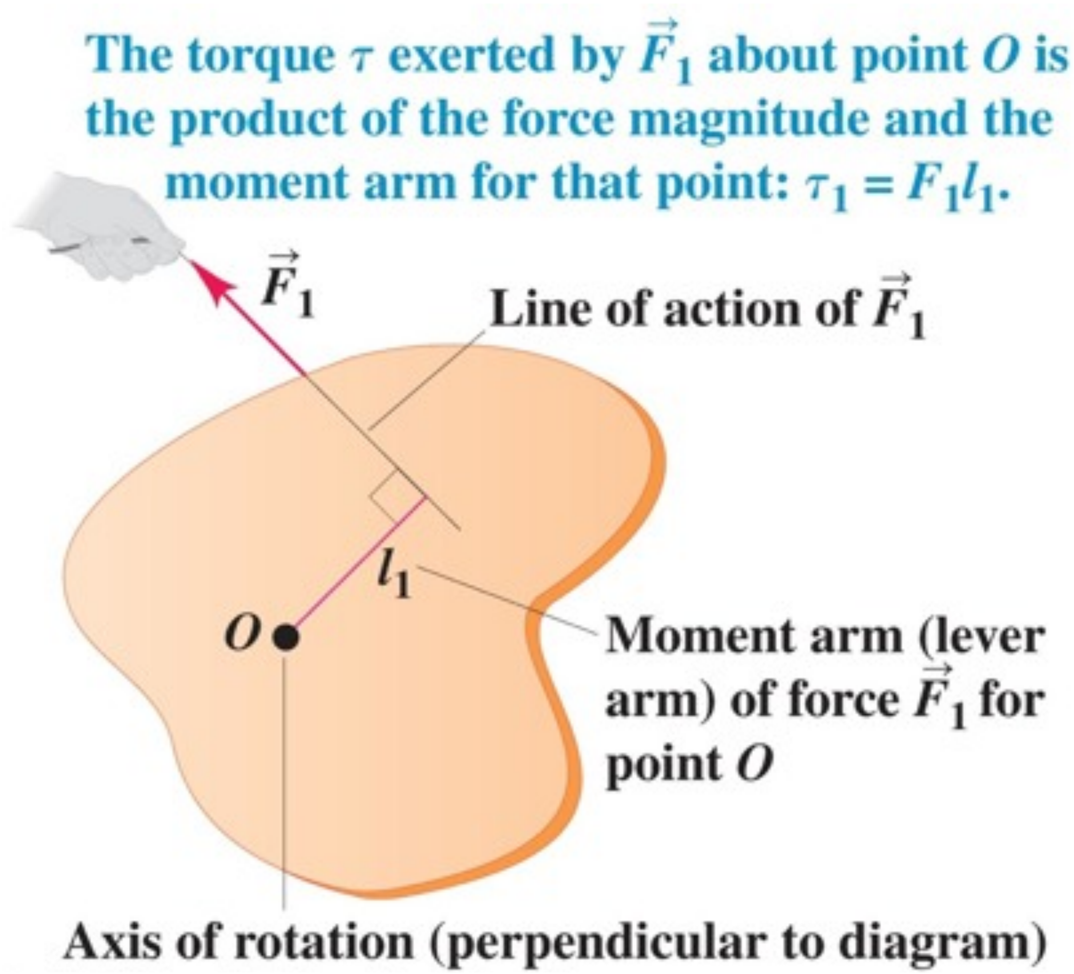
$$v_{\rm cm} = \sqrt{\frac{4}{3}gh}$$

torque

in the same way that a force causes an acceleration by $F = ma$

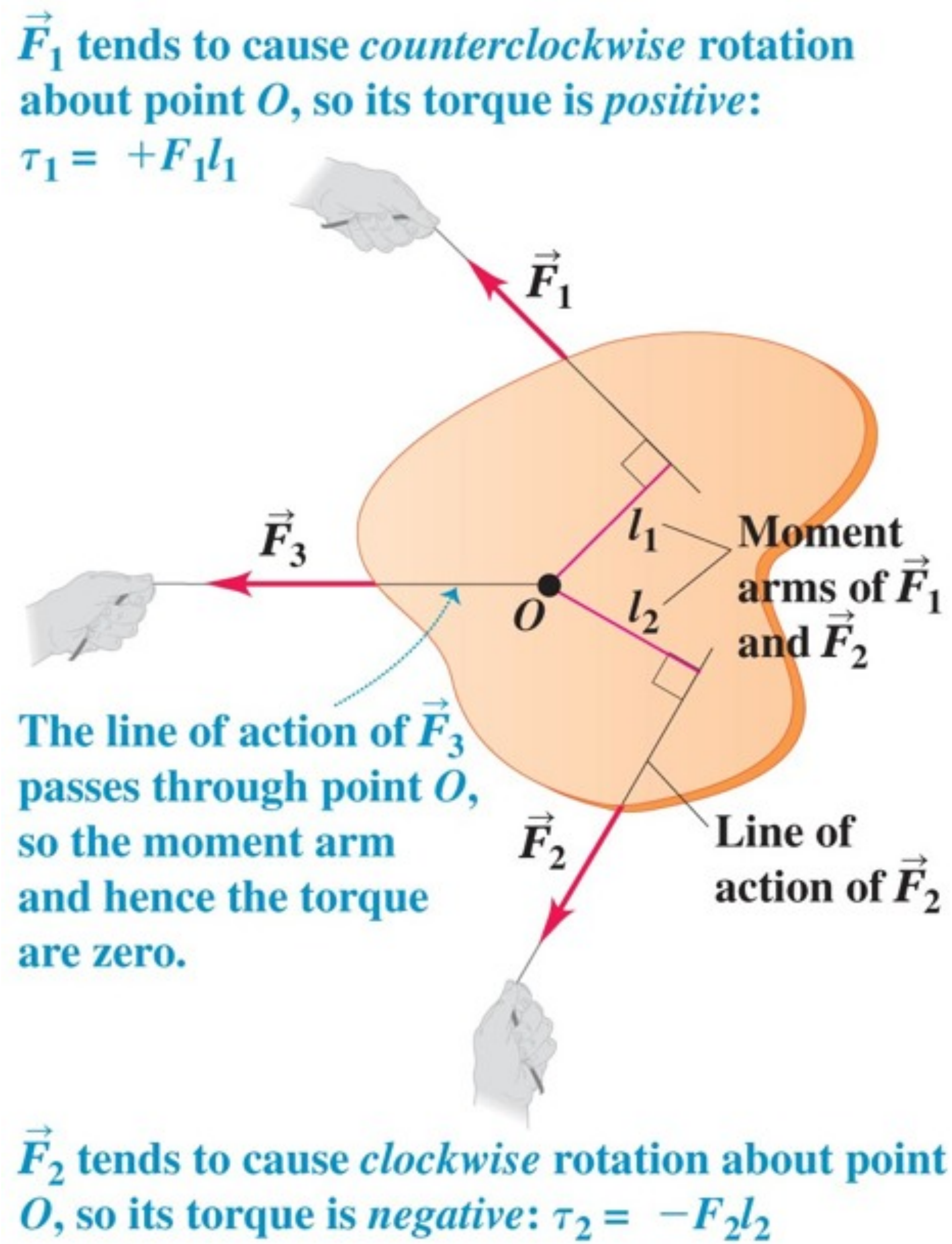
a **torque** causes an **angular acceleration** by $\tau = I\alpha$



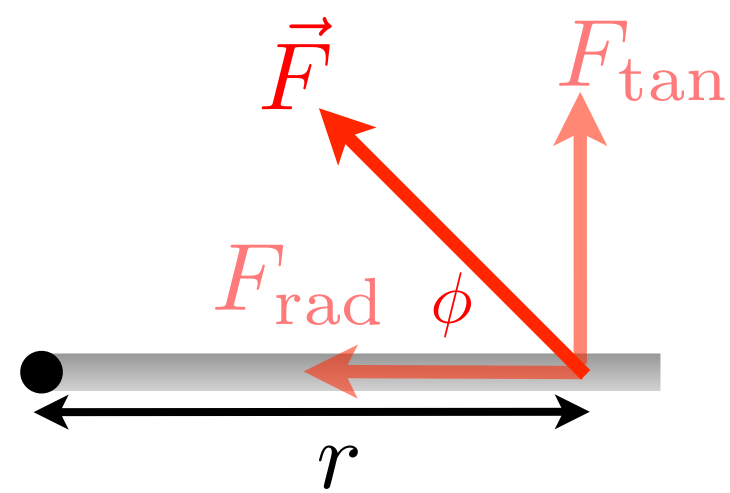


$$\tau = F\ell$$

the sign of torque



only tangential forces provide a torque



$$\tau = F_{\tan} r = F r \sin\phi$$

a useful fact

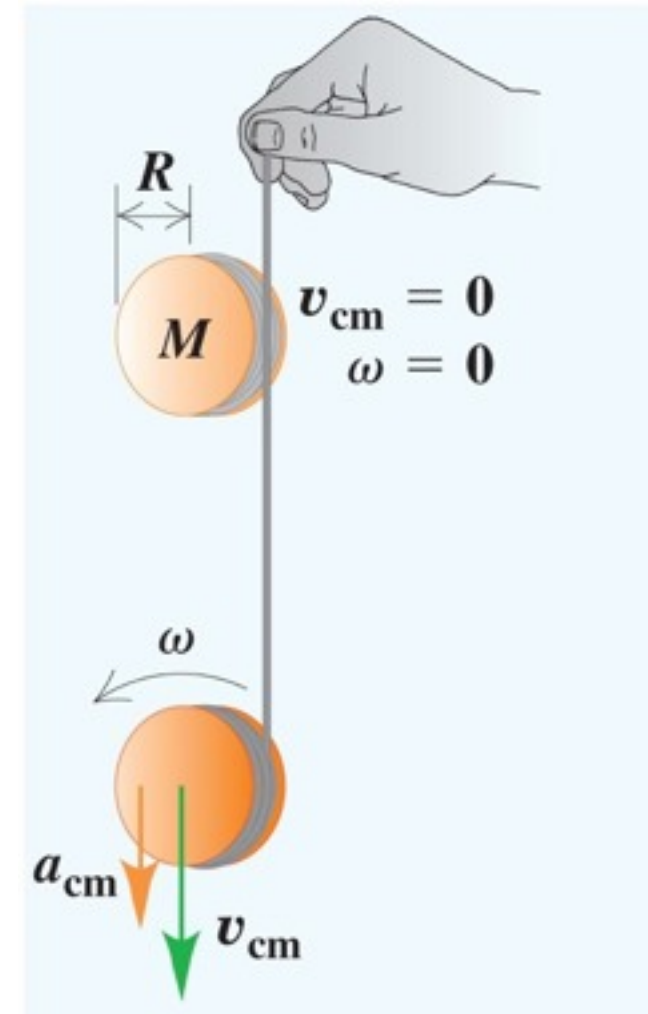
the weight force of a rigid body acts downwards from the center of mass

rotation about a moving axis

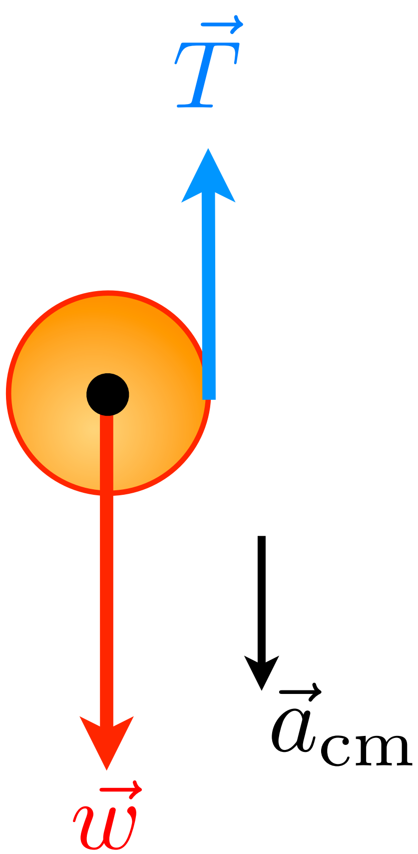
e.g. consider the primitive yo-yo again

let's find the acceleration of the yo-yo & the tension in the string

energy conservation unlikely to help



free-body diagram



Newton's 2nd law for the c.m.

$$w - T = M a_{\rm cm}$$

$$Mg - T = M a_{\rm cm}$$

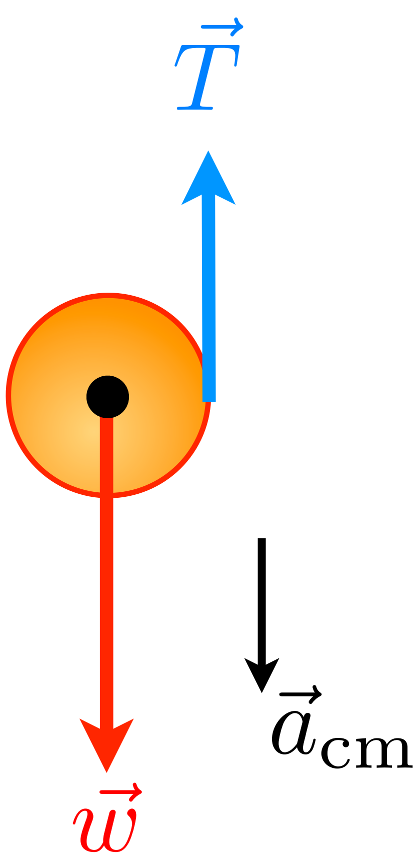
torques & angular acceleration

$$0 + TR = I_{\rm cm} \alpha$$

$$TR = \tfrac{1}{2} M R^2 \alpha$$

rotation about a moving axis

free-body diagram



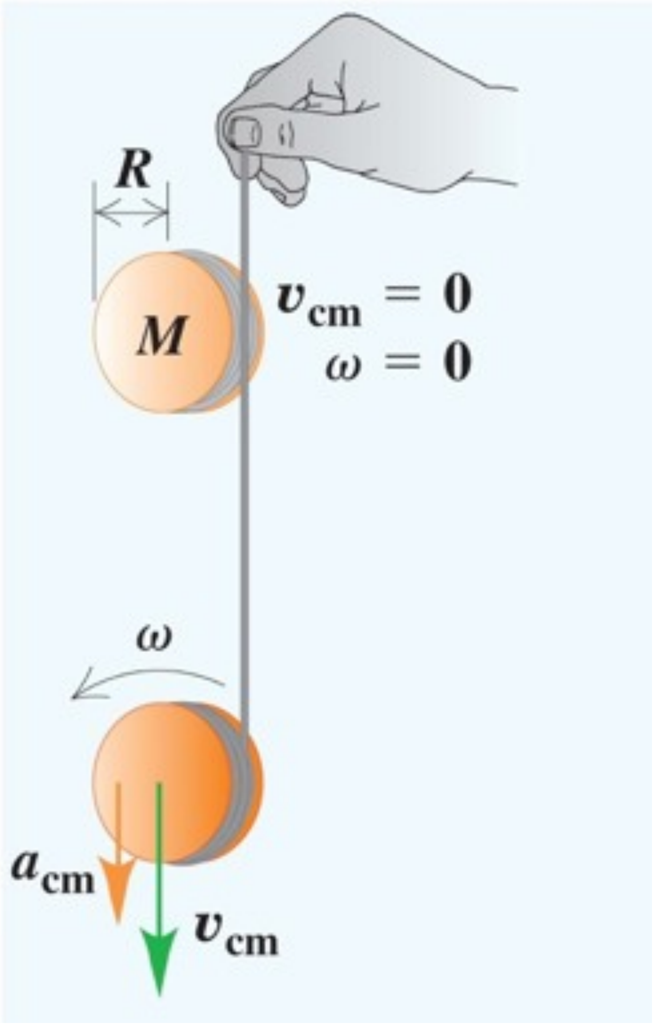
$$Mg - T = Ma_{\rm cm}$$

$$TR = \tfrac{1}{2}MR^2\alpha$$

$$a_{\rm cm} = R\alpha \quad \text{(why ?)}$$

$$T = \tfrac{1}{2}Ma_{\rm cm}$$

$$T = Mg - Ma_{\rm cm}$$



$$a_{\rm cm} = \tfrac{2}{3}g$$

$$T = \tfrac{1}{3}Mg$$

equilibrium of a rigid body

we already know that a body will not remain at rest (or moving with constant velocity) unless

$$\sum \vec{F} = \vec{0}$$

"the net force acting on the body is zero"

clearly a rigid body will not remain non-rotating (or rotating with constant angular speed) unless

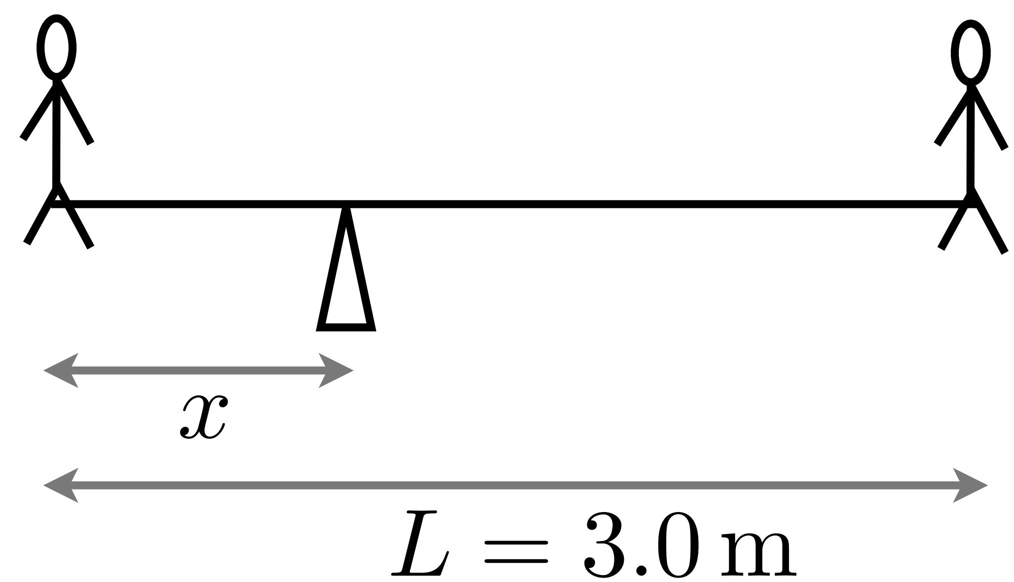
$$\sum \tau = 0$$

"the net torque acting on the body is zero"

if we require a rigid body to remain completely motionless we require **both** these conditions to hold

seesaw

You and a friend play on a seesaw. Your mass is 90 kg and your friend's mass is 60 kg. The seesaw board is 3.0 m long and has negligible mass. Where should the pivot be placed so that the seesaw will balance when you sit on the left end and your friend sits on the right end ?



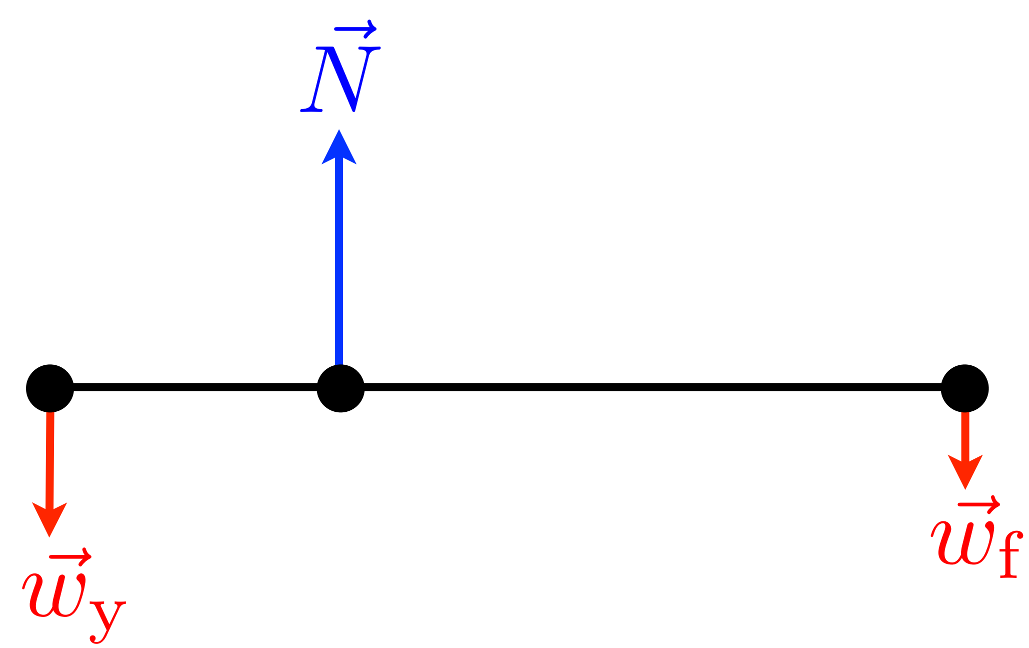
the sum of the torques must be zero if the seesaw isn't to topple

the normal force of the pivot on the board produces no torque

you produce a positive torque of $+m_{\mathrm{y}}g\,x$

your friend produces a negative torque of

$$-m_{\mathrm{f}}g\,(L-x)$$



$$0 = m_{\mathrm{y}}gx - m_{\mathrm{f}}g(L-x)$$

seesaw

You and a friend play on a seesaw. Your mass is 90 kg and your friend's mass is 60 kg. The seesaw board is 3.0 m long and has negligible mass. Where should the pivot be placed so that the seesaw will balance when you sit on the left end and your friend sits on the right end ?

$$0 = m_{\mathrm{y}} g x - m_{\mathrm{f}} g (L - x)$$

$$x = \frac{m_{\mathrm{f}}}{m_{\mathrm{y}} + m_{\mathrm{f}}} L$$

$$\underline{x = 1.2\,\mathrm{m}}$$