Chapter 12 Game Theory: Non-cooperative Games

Game theory is a branch of mathematics. It describes ordinary games and much more. Game theory concerns all situations in which a set of people make choices based on the actual or predicted choices of others. Let’s look at some situations to see whether or not they fall within the purview of game theory.

1. You are deciding whether to take an umbrella to school because it might rain. You have to evaluate the probability of rain and the inconvenience of carrying an umbrella if it does not rain.

2. You are deciding whether or not to take an umbrella to school. One consideration is that you will be able to keep not only yourself dry but also your baby brother, whom you pick up from nursery school on the way home.

3. You are deciding whether or not to bring an umbrella to school. Your boyfriend (or girlfriend) might bring one also. If he brings one, there is no reason for you to bring one also. If he is not going to bring one, you will. You know that he is thinking the same thing. But, you don’t know what he is going to do.

4. A set of people is playing poker.

5. A set of competing firms all offer the same products.

6. A buyer and a seller negotiate about the price of some article for sale.

7. A set of workers in a firm must decide whether or not to join a proposed union.

Now let’s look at whether game theory can be used to shed light on these situations.

1. Certainly you are making a choice. However, game theory will not help you decide what to do. You are not influenced by the choices you think others will make and their choices are not influenced by their anticipation of your choices.

2. You are making a choice, and your choices are affecting whether or not someone else gets wet, but your choices are not affecting his choices and his are not affecting yours. Game theory is not relevant.

3. Here we have a situation in which the choices of two people are influenced by the anticipated choices of the other. This is the subject matter of game theory.

4. This is definitely a situation analyzable with game theory. Poker players make their decisions on the basis of what they think others are going to do.

5. Every firm is influenced by the decisions that other firms make about products, prices, and advertising approaches. This type of situation is amenable to game theoretic analysis.

6. This definitely satisfies the criteria. Both are influenced by the decisions of the other.
There are costs to joining a union; unions have dues and the company may retaliate against union members. Whether or not a worker joins the union may be affected by the choices of other workers. On the one hand, a worker may be less likely to join if he thinks that many others are joining because he feels that his contribution is not necessary - he can enjoy the benefits of higher wages without paying his dues. On the other hand, if many others have joined, the company may be less able to retaliate. This is a situation that can be analyzed using game theory.

Utility

A game involves a set of participants, a set of outcomes that result from the decisions of the participants, and the evaluations that the participants make of these outcomes -- their preferences among these outcomes. We will be assuming that the preferences that individuals have for outcomes can be represented by numbers, and that these numbers can be added and subtracted in a meaningful way. Utility is not the same as money or some objective aspect of the outcome. It is strictly an internal assessment by the individual of the outcome. If \( x \) is an outcome and \( i \) is a player, the utility of player \( i \) for outcome \( x \) is \( U_i(x) \).

Let’s take the following illustration. Ida has absolutely no money or possessions. If she were to earn 100 dollars she would buy some food to satisfy her hunger. Whatever she gets first, it will be what she needs most. A second $100 would inevitably satisfy less pressing needs, and similarly for the third $100. We would say that each additional sum of money brings her less and less additional utility. Suppose, for example, that each doubling of Ida’s fortunes brought her the same additional utility. In other words, the increase in her utility in going from $100 to $200 was the same as going from $200 to $400 and the increase in going from $400 to $800. In other words, it takes more and more additional money to give her the same additional utility.

A linear function, \( U(x) = bx \), where \( b > 0 \) and \( x \) is dollars, would not describe this pattern. For a linear function, each increase in money \( x \) would produce the same increase in utility, \( bx \). Contrary to what we want, going from $100 to $200 would produce an increase of \( 200b - 100b = 100b \), less than the increase of going from $200 to $400, \( 400b - 200b = 200b \). For a linear (straight line) utility function, each additional sum of money would not bring her less and less utility; it would bring her the same increase in utility. And doubling her amount of money would not give her the same increase in utility, but a greater and greater increase. This is not what we want.

However, a logarithm could describe her utility for money. A logarithm (log for short) works because the logarithm of the product of two numbers is the sum of the logs; \( \log(xy) = \log(x) + \log(y) \). Thus, doubling a number always increases its logarithm by the same amount; \( \log(2x) = \log(2) + \log(x) \) regardless of the value of \( x \). Yjis means that doubling Ida’s money will always give her the same increase in utility, \( \log(2) \), regardless of how much money she starts out with.

Equation 1

\[
U_i(x) = K \log(x)
\]
$K$ is an arbitrary constant. Figure 1 shows the relation between $x$ (the amount of money possessed by Ida) and her utility $U(x)$ according to equation 1 and what it would have been if her utility were linearly related to the amount of money she possessed.

![Figure 1](image_url)

As you can see, Ida is happier with more money if $U(x) = \log(x)$, but her happiness increases more and more slowly. This is often (but not always) the case. The first bit of praise received from co-workers, the first promotion, the first kiss -- these are likely to be the most rewarding.

**Expected Utility**

Another assumption we will be making is that decisions are made on the basis of expected utility. To take a simple example, consider placing a bet on a roulette wheel. You are considering two different possible bets. You can bet on a color or on a particular number or not bet at all. If you bet $10 on a color (red or black) you win an additional $10. If you bet on a particular number, you could win $350. Of course, the amount you win is only half the story. The probability of winning if you bet on a color is 18/38, while the probability of winning if you bet on a particular number is only 1/38. How might you combine all these numbers to make a decision? You are drawn in two directions: the
probability of winning is greater if you bet on a color, but the amount you win is greater if you bet on a number. Or, perhaps you are better off if you don’t bet at all!

If you use the principle of expected utility, you would combine probabilities and rewards by multiplying each reward by its probability. Let’s assume that there is no declining utility for money -- each additional dollar is worth the same amount. This contradicts our assumption in the last example with Ida, but might be realistic if small amounts of money are involved. From this point of view, there are three decisions you can make each with its own rewards and probabilities.

<table>
<thead>
<tr>
<th></th>
<th>Winning amount</th>
<th>Winning probability</th>
<th>Losing amount</th>
<th>Losing probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bet on color</td>
<td>$10</td>
<td>18/38</td>
<td>-$10</td>
<td>20/38</td>
</tr>
<tr>
<td>Bet on number</td>
<td>$350</td>
<td>1/38</td>
<td>-$10</td>
<td>37/38</td>
</tr>
<tr>
<td>No bet</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The expected value for betting on a color is ($10)(18/38) - ($10)(20/38) = -$0.53. The expected value for betting on a number is (350)(1/38) - (10)(37/38) = -$0.53. The expected value for not betting at all is (0)(1) - (0)(0) = $0. The expected value of the two bets is equal, but you are better off not placing any bet at all.

More formally, let the vector $x$ be the outcomes of an action and the vector $p(x)$ be the associated probabilities. The expected outcome is the sum of the products of the utilities $x_i$ associated with each outcome times each outcomes probability $p(x_i)$. The expected value is given by the following calculation.

**Equation 2**

$$E(x) = \sum_i x_i p(x_i)$$

Let’s work out another example. Suppose that you are deciding between two investments. With the riskier $1,000 investment, there is a 10% chance of earning $10,000, an 80% chance of earning nothing, and a 10% chance of losing your initial $1,000. With the more conservative investment, there is a 95% chance of earning $300, a 5% chance of earning nothing, and no chance at all of losing your initial investment. Which investment has the higher expected value? For the riskier investment, the expected value is $(10,000)(.10) + (0)(.80) - (1,000)(.10) = $900. For the more conservative investment, the expected gain is $(300)(.95) + (0)(.05) = $285. The riskier investment has the higher expected value. If you used the principal of expected value to make decisions, as we will assume that people do, and if your utility was the same as the money you earned, you would choose the riskier investment.

On the other hand, suppose that your utility was not the same as the amount of money. Suppose it was very important for you to avoid losing the $10,000; this was the rent money. Suppose that the true utility to you of the various outcomes was represented by the following table.

<table>
<thead>
<tr>
<th>$ outcome</th>
<th>True utility</th>
<th>Risky probabilities</th>
<th>Safe probabilities</th>
</tr>
</thead>
</table>

Now you should be able to figure out that if it is very important for you not to lose the $1,000 nest egg, you are better off with the conservative investment.

A strategy

In game theory, a strategy is a complete description of how one would play a game. A strategy includes all possible contingencies. In the examples we have already examined, there are just a few strategies. In the first gambling example, there are just three strategies: bet on a color, a number, or do not bet at all. In the investment decision also, there are just three alternatives: the risky investment, the conservative investment, or no investment at all. In even the simplest real life situations, the number of strategies can be huge because all contingencies must be taken into account. For example, consider the game of tic-tac-toe. Suppose we consider the strategies available to the first player.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
</tbody>
</table>

The cells are numbered from 1 to 9. A strategy should tell you what to do in all possible circumstances. Suppose you are the second player. The first player can make any of nine first moves. For each of these possible moves, your strategy should tell you which of your eight possible moves you should make. Here is one strategy for the second player’s first move.

If he checks cell 1, I will check cell 2
If he checks cell 2, I will check cell 3
If he checks cell 3, I will check cell 4
If he checks cell 4, I will check cell 5
If he checks cell 5, I will check cell 6
If he checks cell 6, I will check cell 7
If he checks cell 7, I will check cell 8
If he checks cell 8, I will check cell 9
If he checks cell 9, I will check cell 1

These nine possibilities can each be answered in eight different ways (you can’t choose the same cell the first player has already chosen). Therefore, there are $8^9 = 134,217,728$ different possible strategies just for your first move, not even considering all
your moves through the game! The number of possible strategies for a complex game like chess or poker is unfathomably large. Despite this, the idea of a strategy is a valuable one.

Strategies describe what the players will do in all possible circumstances. If you know the strategies of all the players in a game, then you could play the game without their active participation. Once the strategies of all the players are known, the outcome is also determined. For example, suppose that the beginning of the first player’s strategy in tic-tac-toe is “Check cell 5 on the first move” and the second player’s strategy is one described above. Then we know what will happen through the first moves: the first player will choose cell 5 and the second player will choose cell 6.

This means that a two-person game can be represented in the form of a matrix in which the rows represent one player’s strategies, the columns represent the second player’s strategies, and the cells describe the outcomes to the two players. For example, take the popular childhood game “Rock, paper, scissors.” The two players simultaneously, through hand signals, show a sign for “rock,” “paper,” or “scissors.” Scissors beats paper, paper beats rock, and rock beats scissors. With respect to any one play of the game, there are just three pure strategies (later we will distinguish between pure and mixed strategies): paper, scissors, and rock. In the following matrix A, the three rows and three columns correspond to the three strategies “rock”, “paper”, and “scissors”. “1” means that the row player wins and “-1” means that the row player loses for that particular combination of row and column strategies. A “0” means that neither wins. The rewards for the column player in matrix B are just the negatives of the rewards for the row players.

Equation 3

\[
A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}
\]

Another, even simpler game is “matching pennies.” Each player simultaneously shows his opponent either side of a coin. It is agreed beforehand that if both sides are the same, one of the players wins, and that if the sides are different, the other player wins. Here the two strategies for one play are “Heads” and “Tails.” The rows and columns of the following matrix refer to the Heads and Tails strategies respectively. The matrices A and B give the rewards for the row player, who wins when the coins show the same side, and the column player, who wins when the coins show different sides.

Equation 4

\[
A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}
\]

Note that the number of strategies in even a simple game like this depends on how many times the game is to be played. If a pair is matching pennies just once, then there are
two strategies. However, suppose that the players are going to match pennies twice. Each player can now tailor his actions on the second move to what his opponent did on the first play. Then, each of the players can choose among the following different strategies:

1. H on the first play. H on the second play if he played H on the first. T on the second play if he played T on the first.
2. H on the first play. H on the second play if he played T on the first. T on the second play if he played H on the first.
3. H on the first play. H on the second play regardless of what he did on the first.
4. H on the first play. T on the second play regardless of what he did on the first.
5. T on the first play. H on the second play if he played H on the first. T on the second play if he played T on the first.
6. T on the first play. H on the second play if he played T on the first. T on the second play if he played H on the first.
7. T on the first play. H on the second play regardless of what he did on the first.
8. T on the first play. T on the second play regardless of what he did on the first.

Each of these is a complete description of a strategy. They dictate what to do under any circumstance. When we know the strategies chosen by the two players, the outcome is determined. We don’t actually need the two players to play the game. For example, suppose that the Row player, who wins if they pick the same face, chooses strategy 3, while the Column player chooses strategy 5. On the first play, Row chooses H and Column T, so Column wins. On the second play, Row plays H and so does Column, so Row wins. Since each won one game, they break even.

Since there are eight strategies, the matrix for this game would have eight rows and columns rather than two (see homework problem 5).

Consider also the following two examples, which we will be using later. In each of them, the matrix shows the rewards to the row player; the reward to the column player is the negative of these numbers.

Equation 5

\[
\begin{pmatrix}
1 & -2 \\ 2 & -1 \\
\end{pmatrix}
\]

Equation 6

\[
\begin{pmatrix}
-2 & 3 \\ 4 & -2 \\
\end{pmatrix}
\]

Two-person zero-sum games

In two-person zero-sum games each player’s rewards are exactly counter-balanced by the other person’s losses. One wins what the other loses. If \(x_1\) and \(x_2\) are their
rewards, it must be true that \( x_1 + x_2 = 0 \) for all outcomes. The Rock, Paper, Scissors game, the Matching Pennies game, and the games described by equations 5 and 6 are examples of zero-sum games. In constant-sum games, the rewards to the two players must add up to a constant that is not necessarily zero. For example, two people dividing up a pie are playing a constant sum game because their shares must add up to 1.00, the whole pie. Clearly, constant-sum games are just as competitive as zero-sum games, and the ideas we will develop apply to both equally.

**Equilibria Using Pure and Mixed Strategies**

What is the best strategy in the game described by equation 5? Would you rather be the row or the column player? It might seem that it does not make any difference, because the average value in the whole matrix is zero.

Let us suppose that the game is played a number of times, so that you can adjust your strategy to the strategy of the column player. Suppose you start by using your first strategy repeatedly. The column player, if she is rational also, will adjust to this by playing her second strategy. The reason is clear. If she plays her first strategy she loses 1 (remember, your gain of 1 is her loss of 1, because the game is zero-sum), but if she plays her second strategy, she wins 2. Now, you are losing 2 for each play of the game. You switch to your second strategy because you lose 1 instead of 2.

If you and your opponent reach this point, there is no further reason for either of you to change. Your second strategy is the best possible response if she plays her second strategy. Moreover, her second strategy is her best response when you play your second strategy (she prefers making 1 to losing 2). There is equilibrium. Neither of you has any incentive to change.

As an exercise (see problem 9 in the homework), see if you can locate the equilibrium pair of strategies in the following game. Each player has three alternative strategies.

**Equation 7**

\[
A = \begin{pmatrix}
3 & -5 & -1 \\
1 & 2 & 0 \\
2 & -1 & -2
\end{pmatrix}
\]

Strategies \( i \) and \( j \) in a two-person zero-sum game described by the matrix \( A \) are in equilibrium if \( a_{ij} \) is the largest element in its column (column \( j \)) and the smallest element in its row (row \( i \)). The reason is simple. If the column player chooses strategy \( j \), the row player can control which outcome in column \( j \) occurs. He has no reason to deviate from the largest element in that column. Similarly, if the row player chooses strategy \( i \), the column player controls which outcome in that row occurs. She has no reason to deviate from the smallest element in that row (which, because the game is zero-sum, is her largest gain).
Now consider the Matching Pennies game. There is no pair of strategies in this matrix that are in equilibrium. For any outcome, one of the players has an incentive to change. If both players show heads or tails, the column player will want to change. If the players show opposite sides of the coin, the row player will want to alter his strategy.

Step back and think about what would be an effective strategy in this game against an opponent as rational as yourself. If you, as the row player, have any tendency to favor heads (or tails), the column player would take advantage of that by showing tails (heads) more frequently. Moreover, if there were any pattern to your play (such as alternating heads and tails), your opponent could detect that pattern and show tails when you were going to show heads and heads when you were going to show tails.

The best way to play this game against a rational opponent, therefore, would be to play both strategies equally often but without any pattern at all. In other words, the best strategy would be to consult a table of random numbers in selecting heads or tails, selecting, say, heads if the random number were even and tails if it were odd. Then your choices would be unexploitable. If you were playing against a good (“rational”) opponent who could be counted on to exploit any weakness in your strategy, it would also be the best strategy to use.

The rows and columns of the matrix are called pure strategies. A random choice from the pure strategies is called a mixed strategy. If there are N pure strategies in a game, then \((p_1, p_2, \ldots, p_N)\), a vector of probabilities summing to 1.00, describes a mixed strategy. Choosing heads and tails with equal probabilities is the mixed strategy \((.50, .50)\). Choosing heads two thirds of the time would be \((2/3, 1/3)\). The pure strategy of always choosing heads would be \((1, 0)\).

The mixed strategy \((.5, .5)\) would not only be best; the combination of both players playing it would also be in equilibrium. If both players were playing this strategy, neither player would have an incentive to change. Suppose that the column player is using \((.5, .5)\) as her strategy. No matter what the row player does, he will average zero. Nothing he can do will increase his expected rewards. The same holds for the column player if the row player is using the \((.50, .50)\) strategy. Therefore, this pair of strategies is in equilibrium.

The mathematicians Oskar Morgenstern and John Von Neumann proved that in any two-person zero-sum game there is an equilibrium pair of strategies, either pure or mixed. An equilibrium pair of pure strategies (if it exists) is easy to find; \(i\) and \(j\) are an equilibrium pair of pure strategies if \(a_{ij}\) is the largest element in its column and the smallest element in its row. That \(a_{ij}\) is the largest in its column means that Row has no incentive to change from strategy \(i\) to a different strategy if Column is playing strategy \(j\). That \(a_{ij}\) is the smallest in its row means that Column has no incentive to change from strategy \(j\) to a different strategy if Row is playing strategy \(i\) (remember that large values of \(a_{ij}\) are bad for Column). Equilibrium pairs of mixed strategies are more difficult to find.

Consider equation 6, for example. There is no pair of pure strategies in equilibrium. In order to solve this situation, we must make use of the fact that the equilibrium pair of strategies for two person zero-sum games is also the best strategy for each player. We must look more closely into what we mean by best. If one is playing an opponent in a zero-sum game who will exploit any weakness, one must assume that she will act in such a way that maximizes her gains (and minimizes your gains) no matter what
strategy you use. Therefore, you should assess your strategies in terms of the worst that can happen to you if you play that strategy (because the worst for you is the best for your opponent). You should then choose a strategy that maximizes your minimum reward because your rational opponent will hold you to your minimum reward. This is the minimax criterion for choosing strategies. It turns out that in two-person zero-sum games, the minimax pair of strategies is also in equilibrium.

Let’s use this fact to solve for the equilibrium pair of strategies in equation 6. Consider the row player. He must decide on a mixed strategy \((\pi, 1-\pi)\), where \(\pi\) is his probability of choosing the first strategy, not 3.1416. When his opponent plays her first pure strategy, his expected reward is \(-2\pi + 4(1-\pi)\). When she plays her second pure strategy, his expected reward is \(3\pi - 2(1-\pi)\). The graph below shows his rewards as a function of \(\pi\). The solid line shows what he expects to earn if his opponent plays the first pure strategy, and the broken line shows what he can expect to earn if his opponent plays the second pure strategy. If his opponent plays a mixed strategy, his expected earnings will be between the two lines.

\[
\begin{align*}
-2\pi + 4(1-\pi) \\
3\pi - 2(1-\pi)
\end{align*}
\]

\(\pi\)

**Figure 2**

The two lines cross when \(\pi = 6/11\). For each value of \(\pi\) the lower of the two lines shows that worst that can happen to him (in terms of his expected payoffs). If \(\pi\) is less than 6/11, the worst happens if the other chooses her second pure strategy. If \(\pi\) is greater than 6/11, the worst happens when the other player chooses her first pure strategy. The lower of the two lines is at its maximum when \(\pi = 6/11\). This is his minimax strategy. The value he earns is given on the vertical axis of figure 2. Where the two lines intersect,
he earns $8/11$ regardless of what the column player does (see problem 8). Similarly, the column player is guaranteed that she will lose no more than $8/11$ if she uses the mixed strategy $(5/11, 6/11)$. This combination of mixed strategies $\{(6/11, 5/11), (5/11, 6/11)\}$, is clearly in equilibrium; neither player will do better by changing his strategy. Finding minimax mixed strategies in games with more than two strategies is more complicated, but the basic ideas remain the same.

Non-zero-sum games

Very few situations, apart from athletic contests, parlor games, and war are purely zero-sum. In most situations there are complicated mixtures of cooperative and competitive interests. Consider, for example, a married couple. With respect to many goals, their interests may be identical. Each wants as high a joint income as possible. Each wants their home to have no necessary expensive repairs. If they have children, they will both want their children to be happy and to do well in school. On the other hand, they may disagree on certain issues. One may want to take skiing vacations while the other wants to spend vacation time on home repairs. Each may not like the other's parents. While both may agree that the children should receive some discipline, each may prefer that the other do it. Both may like fine foods but prefer that the other prepare it.

All the games we will be examining in this chapter are non-cooperative. In game theory this is a technical term meaning that the players cannot arrive at binding agreements concerning their future choices. They may choose to behave generously toward one another, but they cannot be forced to do so. There is no government agency that will punish non-compliance with a contract. There is no outraged community that will wreak its vengeance on a member who violates an agreement. In a cooperative game, such binding agreements are possible.

Let’s look at some classic types of situations that involve mixtures of conflicting and identical interests.

I The Prisoners’ Dilemma

Two prisoners are interviewed separately by the police. Although the police are convinced that they are guilty of a serious crime, they do not have sufficient evidence to convict them. To each of them separately they make the following offer. “If you don’t confess and your partner does, your partner will get off scot-free and you will have the book thrown at you; we will ask for the most lengthy sentence possible, 20 years. If both of you confess, we will ask for the normal prison sentence, 10 years. Even if neither of you confesses, we will get you on some trumped-up charge, like illegal possession of a weapon or parole violation and you will get 2 years in jail. Your partner is being offered the same terms.”

What is the prisoner to do? If his partner has confessed, he is better off confessing. If his partner has not confessed, he is also better off confessing. No matter what his partner does, he is better off confessing. Yet, paradoxically, if both do what is best for themselves, both are worse off than if they behaved against their own interests.
Matrices \( A \) and \( B \) giving the rewards (or punishments) to the two players can represent this game. \( A \) is the matrix for the row player, and \( B \) is the matrix for the column player. In the following matrix, the two rows correspond to the “Don’t Confess” and “Confess” options for one of the prisoners, and the two columns correspond to the “Don’t Confess” and “Confess” options for his partner in crime. The reward matrix \( B \) for the column player is just \( A^t \). Any game in which \( B = A^t \) is called symmetric.

**Equation 8**

\[
A = \begin{pmatrix} -2 & -20 \\ 0 & -10 \end{pmatrix} \quad B = \begin{pmatrix} -2 & 0 \\ -20 & -10 \end{pmatrix}
\]

The essence of the Prisoner’s Dilemma is that if each person acts in his own best interests, both the individuals are worse off. This seems paradoxical. Yet, this potential conflict between the interests of each and the interests of all actually exists in a wide variety of situations. If campers in a wilderness area do not carry out their own trash, the area becomes littered. If listeners to public radio freeload, then the stations may not have enough money to carry on.

There is a more general way to describe the Prisoners’ Dilemma game. There are two strategies: cooperate and do not cooperate. If both players cooperate, they both receive a reward (“\( r \)”). If they both fail to cooperate they receive a punishment (“\( p \)”). On the other hand, there is the temptation (“\( t \)”) not to cooperate if the other does, and there is the risk of receiving the sucker (“\( s \)”) payment by cooperating when one’s partner does not. The dilemma exists because \( t > r > p > s \). The matrices are:

**Equation 9**

\[
A = \begin{pmatrix} r & s \\ t & p \end{pmatrix} \quad B = \begin{pmatrix} r & t \\ s & p \end{pmatrix}
\]

**II Chicken**

In the classic adolescent game of Chicken, two drivers of automobiles drive straight toward one another. The person who changes his course first so as to avoid a collision is the loser -- the “chicken.” The other is the winner. If both change their course, there is no winner. The worst outcome occurs if neither changes his course and a crash occurs. This situation could also be loosely represented by a two by two matrix in which the alternatives are “chicken” and “mule”. Letting winning = 4, surviving as one of two chickens = 3, losing = 2, and dying = 0 represent the utilities of the four outcomes, the following \( A \) matrix represents the interests of the row player. The game is symmetric, so \( B = A^t \).
III The Hawk-Dove Game.

Suppose that members of a species compete over food. When both come upon food simultaneously, they can act tough, like a hawk, or meek, like a dove. Two doves share the food. If a hawk meets a dove, the hawk gets all the food. If two hawks meet, they fight over the food and both risk injury. Let $\rho$ be the value of the food, let $C$ be the cost of injury, and suppose that if two hawks meet, they both have an equal probability $1/2$ of winning and getting the food or losing and being injured; therefore, the expected outcome for two hawks is the average outcome, $(\rho-C)/2$. Also assume that $\rho < C$. This is also a symmetric game which can be represented by the following $A$ matrix (the first row and column refer to the Hawk strategy, the second to the Dove). The payoffs for the column player are given by the matrix $B = A'$.

\[
A = \begin{pmatrix}
3 & 2 \\
4 & 0
\end{pmatrix}
\]

IV The Coordination Game

In the coordination game, two players who cannot communicate with one another profit only if they can make the same choice. Suppose, for example, that two people have agreed to meet for lunch at noon but cannot remember the place they agreed to meet. The two locations may be of unequal value. Let’s also assume that the game is symmetric, so that both players profit equally if they succeed in meeting. Let $a > 0$ and $b > 0$ be the value of the two restaurants. Suppose, moreover, that they have in the past met at both restaurants so that it is not certain that they will meet this time at the better one.

\[
A = \begin{pmatrix}
(a - C)/2 & \rho \\
0 & \rho / 2
\end{pmatrix}
\]

I want to introduce some terms that we will use in analyzing these and other games. Let $u(x, y)$ be the value that someone playing strategy $x$ receives when playing against strategy $y$. $u(x, y) = -u(y, x)$ for all strategies $x$ and $y$ in zero-sum games. Strategy $x$ strictly dominates strategy $y$ if $u(x, z) > u(y, z)$ for all strategies $z$. In the $A$ matrix for the row player, this means that every element in row $x$ is strictly greater than the corresponding element of row $y$. In the Prisoners’ Dilemma, non-cooperation strictly dominates cooperation. Strategy $x$ weakly dominates strategy $y$ if $u(x, z) \geq u(y, z)$ for all
strategies \( z \) and in addition there is at least one \( z \) for which \( u(x, z) > u(y, z) \). In the matrix formulation of a game, this means that every element in row \( x \) is greater than or equal to every element in row \( y \), and at least one element is greater. A strategy is undominated if no strategy weakly dominates it. In the “Paper-Rock-Scissors” game, all strategies are undominated. You can see this by looking at the rows of the matrix in equation 3. The elements of no row are uniformly greater than the elements of any other row. It would seem that it was always best to choose undominated strategies. If \( x \) is dominated by \( y \), then you are never worse off and possibly better off playing \( y \) than \( x \).

**Strategy in Iterated Games**

Thus far we have been mostly considered games that are played just once, or, equivalently, games in which strategies are chosen without regard to what oneself and ones opponent have done in previous plays of the game. This is a realistic assumption when there is only one play of a game, but many games are played repeatedly. For example, people in the same work group or local neighborhood learn which others will remember to repay small loans or return favors and which ones will not. Ted returns the yard equipment he borrows, but Betsy will not without being reminded many times. Those in business learn which associates can be trusted and which ones cannot. That many actions are repeated allows for the development of more complicated strategies that take into account the past behavior of others.

Here are some simple strategies in iterated (repeated) Prisoners’ Dilemma games that have been explored by social scientists.

1. **Tit-for-tat (TFT)**

   The tit-for-tat strategy is to cooperate on the first game with a partner and after that to do what he did in the last game. If he cooperated, tit-for-tat rewards him by cooperating in the next game. If he does not, tit-for-tat punishes him by not cooperating on the next game.

2. **Suspicious tit-for-tat (STFT)**

   Suspicious tit-for-tat is just like tit-for-tat except it does not cooperate in the first game.

3. **Pavlov**

   As in classical learning theory, if Pavlov is rewarded it continues to do the same thing and if it is punished it does something different. Pavlov receives the relatively beneficial \( t \) and \( r \) rewards if the other player cooperates and the lower \( p \) and \( s \) rewards if the other player defects. Therefore, Pavlov repeats its choice (either cooperating or defecting) if its partner cooperated on the last game and changes its choice if its partner defected in the last game.
4. Unconditional cooperation (All-C)

The player using this strategy always cooperates, no matter what the other person does.

5. Unconditional non-cooperation (All-D)

A player using this strategy never cooperates, no matter what the other person does.

The mathematical study of iterated games is simplified if we make an unrealistic but useful assumption. We will assume there is a constant conditional probability \( \rho \) that another interaction will occur and a probability \( 1-\rho \) that no further interaction will occur between any pair. Suppose, for example, that two criminals have cooperated in a successful bank robbery. Suppose the probability is \( 2/3 \) that they will have the opportunity to cooperate again on a bank robbery because there is a \( 1/3 \) probability that one or both of them will be jailed for a long period of time. Therefore, there is a \( 1/3 \) probability that the current arrangement will be their last and a \( 2/3 \) probability that they will be free to cooperate again for a second robbery. If that robbery occurs, there is a \( 1/3 \) probability that it will be their last. Therefore, the probability that a second robbery will occur and that it will be their last is \( 2/3 \times 1/3 \). The second and third columns of the following table show these probabilities. The fourth and fifth columns show the probabilities when the probability that the transactions will continue is \( \rho \).

<table>
<thead>
<tr>
<th>Deal #</th>
<th>Probability of occurring</th>
<th>Probability last deal</th>
<th>Probability of occurring</th>
<th>Probability last deal</th>
</tr>
</thead>
<tbody>
<tr>
<td>current</td>
<td>1</td>
<td>1/3</td>
<td>1</td>
<td>1-( \rho )</td>
</tr>
<tr>
<td>1</td>
<td>( 2/3 )</td>
<td>( 2/3 \times 1/3 )</td>
<td>( \rho )</td>
<td>( \rho (1-\rho) )</td>
</tr>
<tr>
<td>2</td>
<td>( (2/3)^2 )</td>
<td>( (2/3)^2 \times 1/3 )</td>
<td>( \rho^2 )</td>
<td>( \rho^2 (1-\rho) )</td>
</tr>
<tr>
<td>3</td>
<td>( (2/3)^3 )</td>
<td>( (2/3)^3 \times 1/3 )</td>
<td>( \rho^3 )</td>
<td>( \rho^3 (1-\rho) )</td>
</tr>
<tr>
<td>4</td>
<td>( (2/3)^4 )</td>
<td>( (2/3)^4 \times 1/3 )</td>
<td>( \rho^4 )</td>
<td>( \rho^4 (1-\rho) )</td>
</tr>
</tbody>
</table>

Now let us suppose that each bank robbery is worth $10,000 to each of them. Using equation 2, the expected value of all their transactions is \( 10,000 \times 1 + 10,000 \times 2/3 + 10,000 \times (2/3)^2 + \ldots \). More generally, if the utility received by a player in game \( i \) is \( u_i \), then his expected utility, \( U \), over all the games is clearly:

**Equation 13**

\[
E(U) = \sum_{i=0}^{\infty} \rho^i u_i
\]
If a player’s rewards in every game are the same ($u_i = u$), then equation 13 has an especially simple form. $E(U) = u + pu + \rho^2u + \rho^3u + \ldots = u(1 + \rho + \rho^2 + \rho^3 + \ldots)$. The infinite sum in parentheses turns out to be equal to $1/(1 - \rho)$ when $|\rho| < 1$, which it is because $\rho$ is a probability. In the last example, with the two bank robbers, $u = $10,000 and $\rho = 2/3$, so their expected gain for collaborative bank robberies was $10,000/(1-2/3) = $30,000.

Consider, as another example, two players in a Prisoners’ Dilemma, each of which has the choice between playing tit-for-tat, or always defecting, or always cooperating (these are the only three strategies that occur to them or that are available to them). There are six different combinations of the three strategies that we must consider.

1. Tit-for-tat versus tit-for-tat

Both players cooperate earning $r$ in the first game (please refer to equation 9). With a probability $\rho$, they earn $r$ in the second trial, with a probability $\rho^2$ they earn $r$ in the third game, and so on. Their total expected reward is $r + \rho r + \rho^2r + \rho^3r + \ldots = r(1 + \rho + \rho^2 + \rho^3 + \ldots) = r/(1 - \rho)$.

2. All-C versus All-C

Here also the expected reward for both players is $r/(1 - \rho)$.

3. All-D versus All-D

In this case both players earn $p/(1 - \rho)$ because they never cooperate.

4. Tit-for-tat versus All-C

Both players always cooperate and they both earn $r/(1 - \rho)$.

5. Tit-for-tat versus All-D

In the first game, TFT earns $s$ and All-D earns $t$. In every succeeding game, both earn $p$. Therefore, the total rewards are $s + \rho p/(1 - \rho)$ and $t + \rho p/(1 - \rho)$ respectively for TFT and All-D (see homework problem 11).

6. All-C versus All-D

The expected gain for the two strategies is $s/(1 - \rho)$ and $t/(1 - \rho)$ respectively.

We can put all this in the form of a matrix $A$ in which the three rows and columns correspond to the tit-for-tat, unconditional cooperation, and unconditional non-cooperation respectively. The values of $A$ show the gains for the row player. The matrix $B = A^T$ gives the gains for the column player.
Equation 14

\[
A = \begin{pmatrix}
  r/(1-p) & r/(1-p) & s + pp/(1-p) \\
  r/(1-p) & r/(1-p) & s/(1-p) \\
  t + pp/(1-p) & t/(1-p) & p/(1-p)
\end{pmatrix}
\]

The Nash Equilibrium for Pairs of Strategies in a Non-Cooperative Game

Consider the coordination game (equation 12). This is not a zero-sum game, but it has equilibria. If both players were to choose their first strategy, neither one has a unilateral incentive to change. Suppose, for example, that the row player intended to show up at the first restaurant and before he left home he found a note from the second player saying that he too intended to show up at the first restaurant. This information would not lead the first player to change his mind. On the other hand, suppose that the note said that the second player intended to show up at the second restaurant. This would lead him to change his plans. Two of the four pairs of strategies in equation (12) have this kind of stability to them: \{1, 1\} and \{2, 2\}, have this stability. These strategies are in Nash equilibrium.

Definition. Let the matrices \(A\) and \(B\) represent the rewards for the row and column players in a game and let \(v_1(\alpha, \beta)\) and \(v_2(\alpha, \beta)\) be their rewards if they choose strategies \(\alpha\) and \(\beta\) respectively. The pair of strategies \(\{\alpha, \beta\}\) for the row and column players are in Nash equilibrium if each is the best response to the other. More specifically, \(v_1(\alpha, \beta) \geq v_1(\phi, \beta)\) for any other strategy \(\phi\) available to the row player, and \(v_2(\alpha, \beta) \geq v_2(\alpha, \theta)\) for any other strategy \(\theta\) available to the column player.

It is obvious that mutual non-cooperation is in equilibrium in the Prisoners’ Dilemma game. In the Hawk-Dove and Chicken games, there would appear to be two Nash equilibria: (Hawk, Dove) and (Dove, Hawk) in the Hawk-Dove game and (Chicken, Mule) and (Mule, Chicken) in the Chicken game (see homework problems 14 and 15). In the Rock-Paper-Scissors game it would appear that there are no Nash equilibria. It turns out that all but the first of these statements are wrong when we consider not just pure strategies but mixed strategies as well. With mixed strategies, the Hawk-Dove and Chicken games have three, not two, Nash equilibria, and the Rock-Paper-Scissors game does have a Nash equilibrium. In fact, every game of the sort we have been considering has at least one Nash equilibrium when mixed strategies are taken into account.

Let us first consider the game of Chicken (Equation 12). Let the two opponents be Row and Column. It is obvious that if either player is known to be committed to the Mule strategy, it is a Nash equilibrium for the other to choose the Chicken strategy; being a Mule too means death. These are the two Nash equilibria among the pure strategies. Now let us look for Nash equilibria among the mixed strategies. Row’s mixed strategy is the probability \(p_1\) that she will choose the Chicken strategy. Column’s mixed strategy is the probability \(p_2\) that she will use the Chicken strategy. Row’s best response to Column’s strategy depends on what strategy Column has chosen. Multiplying her rewards by
Column’s probabilities of choosing Chicken and Mule, Row’s expected rewards are $3p_2 + 2(1-p_2)$ if she chooses to be a Chicken and $4p_2$ is she chooses to be a Mule. She is better off as a Chicken if $p_2 < 2/3$ and better off as a Mule if $p_2 > 2/3$. If $p_2 = 2/3$, it makes no difference what strategy she chooses; her rewards are 8/3 regardless. The game is symmetric, so the same criterion holds for Column. Column’s expected rewards are $3p_1 + 2(1-p_1)$ if she chooses to be a Chicken and $4p_1$ is she chooses to be a Mule. She is better off as a Chicken if $p_1 < 2/3$ and better off as a Mule if $p_1 > 2/3$. If $p_1 = 2/3$, it makes no difference what strategy she chooses.

The following graph plots Row’s and Column’s best responses.

The dotted line shows Column’s best responses to Row’s choice of $p_1$, the broken line shows Row’s best responses to Column’s choice of $p_2$. Where the two lines cross, each person’s strategy is the best response to the other’s strategy. This occurs in three places: $\{(1,0), (0,1)\}$, $\{(0,1), (1,0)\}$, and $\{(2/3,1/3), (2/3,1/3)\}$. These are the three Nash equilibria for this game.
Note that these are completely different from the minimax solution, which would be for both players to choose chicken (by avoiding death, this would maximize their minimum gains). In non-zero-sum games, the pair of minimax solutions will not in general be in equilibrium, and that is certainly true in this case.

**The Evolution of Cooperation - Axelrod's Famous Tournament**

One would think that tit-for-tat would be a fairly successful strategy, despite its tremendous simplicity. It encourages cooperation by cooperating on the first game. It rewards cooperation and punishes non-cooperation in a consistent way. At the same time, it seems reasonable that more complicated and subtle strategies could do even better in iterated Prisoners’ Dilemmas. Tit-for-tat does not take advantage of even more trusting strategies like Unconditional Cooperation. A better strategy might be to throw in an occasional non-cooperative choice to see if the other person responds; if the other continues to cooperate, it is safe to exploit him.

In the late 1970s Robert Axelrod invited a number of game social scientists and game theorists to submit strategies in the form of computer programs for the iterated Prisoners Dilemma game. Fourteen strategies were submitted. A computer round robin tournament was arranged in which each strategy met every other strategy. The winner in terms of total points accumulated was Tit-for-tat, submitted by the eminent game theorist Anatol Rapoport. Axelrod published all of the results from his tournament and invited more participants to submit strategies. He received sixty-two entrants from social scientists, mathematicians, biologists, physicists, and computer hobbyists. Tit-for-tat was again submitted, by Anatol Rapoport, and again it won.

Tit-for-tat was successful not because it beat other strategies in head-to-head meetings. In fact, it never did better than any of its partners. For example, when All-D meets Tit-for-tat, they earn $t$ and $s$ respectively on the first game and then they both equally earn $p$ on every successive game. Since $t > s$, Tit-for-tat never catches up. Tit-for-tat does better than All-D over all because it resists exploitation, it cooperates with cooperators, and, by cooperating in the first game, it encourages mutual cooperation.

**Collective Stability and Evolutionary Stability**

The success of the cooperative Tit-for-tat in the tournament suggests that it is not true that “nice guys always finish last.” It also suggests a solution to an old biological and sociological puzzle - how does cooperation evolve in humans and in animals when cheaters often do better? In terms of biology, animals programmed with successful strategies will leave more progeny and will exert more influence of the evolutionary progress of a species. If cheating strategies often win out over more cooperative strategies, how can cooperation evolve? Axelrod and a biologist, William Hamilton, developed some mathematics to describe characteristics of successful strategies.

Consider, for example, the three strategies in equation 13: Tit-for-tat (TFT), Unconditional Cooperation (All-C) and Unconditional Non-cooperation (All-D). We are to imagine a world in which pairs of organisms interact randomly with one another, that interactions have consequences for the health and success of the organisms, and that they
may interact more than once ($\rho$ is the probability that a pair of interacting organisms will interact again). The organisms could be females in a troop looking after each others' offspring. Cooperating would mean protecting the offspring of other mothers with as much diligence as one protects ones own. Non-cooperation would be attending almost exclusively to ones own progeny. Or, they could be business firms collaborating on projects. Cooperation would be keeping the terms of the agreement, and non-cooperation could be finding legal ways to exploit one's partner to ones own advantage.

To illustrate, suppose that $t = 2$, $r = 1$, $p = 0$, and $s = -1$. The matrix in equation 14 then becomes:

**Equation 15**

$$
\begin{pmatrix}
1/(1-\rho) & 1/(1-\rho) & -1 \\
1/(1-\rho) & 1/(1-\rho) & -1/(1-\rho) \\
2 & 2/(1-\rho) & 0
\end{pmatrix}
$$

The rows represent the TFT, All-C, and All-D strategies. If $\rho < \frac{1}{2}$, All-D dominates TFT and All-C. If, on the other hand, $\rho \geq \frac{1}{2}$, TFT is in Nash equilibrium with itself with respect to these three strategies (it is its own best response). When a strategy is in Nash equilibrium with itself, we say that it is collectively stable.

Consider the evolutionary implications of collective stability. Suppose that a population consists entirely of organisms using one strategy. Now suppose that there is a mutation, or an invasion from another population. The invaders will be small in relation to the existing population. If the population has been following a collectively stable strategy, then the invaders, who will almost exclusively be interacting with members of the old population, will not flourish because they will not be doing any better, and possibly worse, than members of the old population. Suppose, for example, that $\rho = \frac{3}{4}$ and a population of organisms using TFT is invaded by a small number of All-D users. The All-D users do not flourish because they do less well against the TFT users (earning 2) than the TFT users do against one another (earning 4). If $\epsilon$ (very small) is the proportion of All-D invaders, then $\epsilon$ will also be the proportion of every organism’s interactions with the invaders. TFT users will average $4(1-\epsilon) - \epsilon$ in reward and All-Ds will average $2(1-\epsilon)$. When $\epsilon$ is very small, the former is greater than the latter (see homework problem 16). The invaders will not thrive. It can be proven that TFT is collectively stable against any strategy (not just All-D) if $\rho$ is sufficiently large.

However, the concept of collective stability is not strong enough to guarantee evolutionary stability in the long run. Suppose, for example, that a population consists entirely of organisms using TFT. Suppose that this population is invaded by organisms using the All-C strategy. TFT does not do any better against itself than All-D does. All-C users do just as well as TFT users, and there is nothing to prevent them from increasing through a kind of random drift. But, if there are enough All-C organisms present, then the population is vulnerable to invasion by the ruthless All-D strategy, which can feed off the All-C strategy.
Let’s work out an example. We will assume, as before, that \( t = 2, r = 1, p = 0, \) and \( s = -1. \) Let \( \rho = .65. \) Moreover, suppose that 2/3 of the population is TFT and 1/3 consists of recently invading All-D organisms. The TFT strategies earn 1.57 on the average, while the All-D strategies earn only 1.33 (see homework problem 19). The following graph shows what might happen over time if organisms with more successful strategies have more progeny. The solid line shows the proportions of TFT strategizers while the broken line shows the proportion of All-D players. The TFT population successfully resists the invasion of All-D players, who disappear over time.

Now suppose, as before, that there are 1/3 invading All-D players, but that the TFT population has “drifted” so that ¼ of it is All-C. Now we are looking at a population of 50% TFT, 17% All-C, and, as before, 33% All-D.
The pattern is quite different. This population is vulnerable to invasion. The bottom broken line shows that the All-C strategies disappear, but not before the All-D strategies have fattened up enough to dominate and become the new prevalent strategy. It was results like this that led biologists and social scientists to search for a stricter criterion for resistance to invasion. The concept they developed was evolutionary stability. Let \( u(\alpha, \beta) \) be the payoff to someone using strategy \( \alpha \) if their opponent is using strategy \( \beta \).

**Definition:** Strategy \( \alpha \) is *evolutionarily stable* if either one of the following two conditions holds:

1. \( u(\alpha, \alpha) > u(\beta, \alpha) \) for every other strategy \( \beta \), or
2. \( u(\alpha, \alpha) \geq u(\beta, \alpha) \) for every other strategy \( \beta \), and if \( u(\alpha, \alpha) = u(\beta, \alpha) \), then \( u(\alpha, \beta) > u(\beta, \beta) \)

The first statement says that \( \alpha \) is an evolutionary stable strategy (ESS) if it is strictly better against itself than any other strategy. The second condition says that if some other strategy \( \beta \) is as good against \( \alpha \) as \( \alpha \) is itself, then \( \alpha \) must be better against \( \beta \) than \( \beta \) is against itself. Consider TFT. It does not satisfy the first criterion: it is not better than All-C against itself. It does not satisfy the second property either because it is not better than All-C against All-C. Hence, TFT is, as we have seen, subject to invasion. It can accumulate other strategies that are more vulnerable than itself and provide the food for other invading strategies, like All-D.

Let’s work out an example of an ESS strategy. Consider the Chicken game. There are, as we have seen, three Nash equilibria: \( \{(1,0), (0, 1)\} \), \( \{(0, 1), (1, 0)\} \), and \( \{(2/3, 1/3), (2/3, 1/3)\} \). Only \( \{(2/3, 1/3), (2/3, 1/3)\} \) is a best response to itself. If both play this strategy, each earns 24/9 = 8/3. If Column plays the mixed strategy \( (2/3, 1/3) \), any row strategy is just as good as \( (2/3, 1/3) \) for Row; he continues to earn 8/9 as long as Column plays \( (2/3, 1/3) \). If \( \alpha = (2/3, 1/3) \), we have shown that \( u(\beta, \alpha) = u(\alpha, \alpha) \), where \( \beta \neq (2/3, 1/3) \), so this strategy does not satisfy the first part of the definition of Evolutionary
Stability. If \((2/3, 1/3)\) is to be ESS, then it must be true in addition that \(u(\alpha, \beta) > u(\beta, \beta)\) when \(\beta \neq (2/3, 1/3)\). To show this, let \(\beta = (p, 1-p)\). Then:

**Equation 16**

\[
u(\alpha, \beta) = 3p(2/3) + 2(1-p)(2/3) + 4p(1/3) + 0(1-p)(1/3)\]

**Equation 17**

\[
u(\beta, \beta) = 3p^2 + 2p(1-p) + 4p(1-p) + 0(1-p)^2\]

It is not too difficult to show (see homework problem 20) that \(u(\alpha, \beta) > u(\beta, \beta)\) whenever \(\beta \neq (2/3, 1/3)\). Thus, \((2/3, 1/3)\) is an ESS strategy.
Homework

1. Could the following situations involve the use of game theory?
   a. playing roulette
   b. buying shares in the stock market
   c. as the captain of a ship, deciding when to launch depth charges against an enemy submarine
   d. deciding when to launch the next space shuttle

2. Suppose that Paul’s utility for praise from his co-workers is a log function of the amount of praise he receives. Suppose that his utility is 1.00 if he is praised 10 times. How much utility will he receive by being praised 50 times? 100 times? 500 times? Now find these same quantities if his utility were linear with the amount of praise he received.

3. Suppose that Betty’s utility for money is logarithmic. Betty can buy stock that will earn her $50 for certain or buy another stock that has a 50% chance of earning her $10 and a 50% chance of earning her $90. Which stock does she buy?

4. Calculate the expected value of the following lottery tickets. Assume that utility is linear with money.
   a. .000001 probability of winning 1 million dollars.
   b. .000001 probability of winning a million dollars and .00001 chance of winning $10,000
   c. .05 chance of winning $100 and .5 chance of winning $5.

5. For the game of matching pennies played twice, we saw that there were eight strategies. Fill in all the rewards for the row player in this matrix.

6. Describe a strategy for tick-tack-toe (it may be a stupid strategy).

7. Describe the equilibrium strategies for both players for equation 5.

8. Show that if the row player in equation 6 plays his minimax strategy, he will expect to earn 8/11.

9a. Show that for a 2-person non-cooperative, zero-sum game, a minimax strategy is always a Nash-equilibrium strategy, and visa versa.

10. One strategy, α, is said to strictly dominate another, β, if \( u(\alpha, \tau) > u(\beta, \tau) \) for all strategies \( \tau \) available to the other player. \( \alpha \) is said to weakly dominate \( \beta \) if \( u(\alpha, \tau) \geq u(\beta, \tau) \) for all strategies \( \tau \) and \( u(\alpha, \tau) > u(\beta, \tau) \) for at least one \( \tau \). Consider Chicken, the Prisoners’ Dilemma, the Hawk-Dove game, the Coordination game, and Scissors-
Paper-Rock. In these five games, does any strategy weakly dominate another? Does any strategy strictly dominate another?

11. Verify all the values in equation 14. Show your work.

12. Describe a strategy in the iterated Prisoners’ Dilemma game that is different from any of those you have read. Hint: your strategy might use the last two or three moves of its opponent instead of only the last move.

13. In the Coordination game (equation 12), find the Nash equilibria.

14. In the Hawk-Dove Game (equation 11), find the Nash equilibria among all the mixed and pure strategies if $C > \rho$.

15. In the Hawk-Dove game, find the Nash equilibria among all the mixed and pure strategies if $\rho > C$.

16. Suppose that in a symmetric game, one strategy weakly dominates all others. Is it necessarily in Nash equilibrium with itself? Why, or why not?

16a. Show that a pair of strongly dominant strategies in a two-person non-cooperative game is always the unique Nash equilibrium strategy pair.

16b. A non-cooperative game with three pure strategies has the payoff matrix

\[
A = \begin{pmatrix}
\rho & 1 & -1 \\
-\rho & 1 & \\
-1 & -\rho & \\
\end{pmatrix}
\]

where $|D|$ is much smaller than 1. Find all Nash equilibrium strategies when $D > 0$ and when $D < 0$.

17. In equation 15, verify that TFT is collectively stable against these two alternatives when $1 > \rho \geq \frac{1}{2}$. Is All-C collectively stable when $1 > \rho \geq \frac{1}{2}$?

17a. Show that an ESS strategy is always a Nash-equilibrium strategy.

18. Using equation 15, show that if $\epsilon$ is the proportion of All-D players, $1-\epsilon$ the proportion of TFT players, and $\rho = 3/4$, TFT will average $4(1-\epsilon) - \epsilon$ and All-Ds $2(1-\epsilon)$. Show that TFT players will do better if they than All-Ds as long as they are more than one-third of the population.
19. Show that if $\rho = 0.65$ and 2/3 of the population is TFT and 1/3 All-D, TFT will average 1.57 while All-D averages 1.33.

19. Using equations 16 and 17, show that $u(\alpha, \alpha) > u(\alpha, \beta)$ whenever $\beta \neq (2/3, 1/3)$

21. In the Hawk-Dove Game, what are the ESS strategies?

22. Find all the ESS strategies for the PD (non-iterated).

23. Show that $(1/3, 1/3, 1/3)$ in which each pure strategy is played with the same probability is an ESS if $\lambda > 0$ but that there is no ESS if $\lambda < 0$.

24. Consider the symmetric game with payoff matrix

\[
A = \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\]

where $a_{11} > a_{12} > a_{11} > a_{21}$.

Show that the only ESS strategies are the two pure strategies.

25. Show that when $\rho > (t-r)/(t-p)$, TFT is ESS against All-D.
Answers to Homework

1. a. No. There is no opponent making choices.
   b. Yes. Others’ choices and your own all affect the price of the stock.
   c. Yes. You put the depth charges where you think the submarine captain put his ship, and he tries to put his ship where he thinks you will not drop depth charges.
   d. No. There is no opponent.

2. \( U(x) = K \cdot \log(x) \), by equation 1. \( U(10) = 1 = K \cdot \log(10) = 1 \), so \( K = 1 \). Therefore, \( U(50) = 1 \cdot \log(50) = 1.70 \).

3. With respect to the conservative option, \( U(50) = 1.70 \); she is certain of this much utility. \( U(10) = 1 \) and \( U(90) = 1.954 \). So, if she takes the risky alternative, her expected utility by equation 1 is \( (1/2)1 + (1/2)1.954 = 1.477 \). Therefore, she should pick the conservative alternative.

4. a. \((.000001) \cdot (1,000,000) = 1\)
   b. \((.000001) \cdot (1,000,000) + (.00001) \cdot (10,000) = 1.1\)
   c. \((.05) \cdot (100) + (.5) \cdot (5) = 7.5\)

5. Let \((X|YZ)\) mean that a player plays \(X\) on the first move, and on the second move plays \(Y\) if his opponent has played \(H\) and \(Z\) if his opponent has played \(T\) on the first move. \(X, Y, \) and \(Z\) can all take the values \(H\) or \(T\). The row person wins if the coins show the same face, the column person if the coins show different faces.

|     | H|HH | H|HT | H|TH | H|TT | T|HH | T|HT | T|TH | T|TT |
|-----|----|----|----|----|----|----|----|----|----|----|----|----|
| H|HH | 2  | 2  | 0  | 0  | 0  | 0  | -2 | -2 |
| H|HT | 2  | 2  | 0  | 0  | -2 | -2 | 0  | 0  |
| H|TH | 0  | 0  | 2  | 2  | 0  | 0  | -2 | -2 |
| H|TT | 0  | 0  | 2  | 2  | -2 | -2 | 0  | 0  |
| T|HH | 0  | -2 | 0  | 2  | 0  | 2  | 0  | 0  |
| T|HT | 0  | -2 | 0  | -2 | 0  | 2  | 0  | 2  |
| T|TH | -2 | 0  | -2 | 0  | 2  | 0  | 2  | 0  |
| T|TT | -2 | 0  | -2 | 0  | 2  | 0  | 2  | 0  |

6. Example - Do the opposite to what the other person did in the last game.

7. The element in the second row and first column, 2, is the largest in its column and the smallest in its row. Therefore, the equilibrium is for Row to play strategy 2 and for column to play strategy 1.

8. It was shown that Row's minimax strategy was to play strategy 1 with a probability of 6/11. If he does so, his expected gain when Column plays strategy 1 is \(-2(6/11) + 4(5/11) = 8/11\). This is also his expected gain when Column plays his strategy 2:

\[
3(6/11) - 2(5/11) = 8/11
\]

9. The element \(a_{23} = 0\) is the largest its column and the smallest in its row. Therefore, the equilibrium pair of strategies is for Row to play 2 while Column plays 3.

10. In Chicken, the Coordination game, and Scissors-paper-Rock, no strategy weakly or strongly dominates another. In the Hawk-Dove game, Hawk strongly dominates Dove.
if \( \rho > C \) and weakly dominates Dove if \( \rho = C \). In the PD game, non-cooperation strongly dominates cooperation.

11. When TFT or All-C meet, the result is complete cooperation. All earn \( r + \rho r + \rho^2 r + \rho^3 r + \ldots = r/(1 - \rho) \). When All-D meets All-D, the result for all is \( p + \rho p + \rho^2 p + \rho^3 p + \ldots = p/(1 - \rho) \). When All-C meets All-D, All-C earns \( s \) in every game for an expected payoff of \( s + \rho s + \rho^2 s + \rho^3 s + \ldots = s/(1 - \rho) \). When All-D meets All-C, it earns \( t + \rho t + \rho^2 t + \rho^3 t + \ldots = t/(1 - \rho) \). When TFT meets All-D it earns \( s + \rho p + \rho^2 p + \rho^3 p + \ldots = s + \rho p + (1 + \rho + \rho^2 + \ldots) = s + \rho p/(1 - \rho) \).

12. An example: Cooperate only if half or more of one’s opponent’s past moves have been cooperative.

13. Let \( p_1 \) be the probability that Row plays its first strategy and let \( p_2 \) be the probability that column plays its first strategy. By constructing a “best response” graph similar to Figure 3, it is clear that Row’s best response is 1 when \( p_2 \geq b/(1 - a) \) and Column’s best response is strategy 1 when \( p_1 \geq b/(1 - a) \). The best response lines cross at three points: where \( p_1 = p_2 = 1 \), where \( p_1 = p_2 = 0 \), and where \( p_1 = p_2 = b/(a + b) \). When Row plays this last strategy, Column earns the same whether Column plays its first or second strategy. Therefore, Row’s strategy \( p_1 \) must satisfy the following equation:

\[
(1 - p_1) = 0 p_1 + b/(1 - p_1).
\]

14. Let \( p_1 \) and \( p_2 \) be the probabilities that Row and Column play the Hawk strategies. If \( C > \rho \), neither strategy dominates the other. Drawing a best-response graph similar to Figure 3, the two best responses intersect at three points: where \( p_1 = 0 \) and \( p_2 = 1 \), where \( p_1 = 1 \) and \( p_2 = 0 \), and where \( p_2 = \rho(C - \rho)/2 + (1 - p_2)\rho = 0 p_2 + (1 - p_2)\rho/2 \). This equation implies that \( p_2 = \rho/C \).

15. If \( \rho > C \), then the Hawk strategy dominates the Dove strategy. The only Nash equilibrium is for both to play the hawk strategy.

16. Let strategy \( i \) dominate all others. This means that for every \( k \neq i \), \( a_{ij} \geq a_{kj} \) for all \( j \) and \( a_{ij} > a_{kj} \) for some \( j \). When \( j = i \), this means that \( a_{ii} \geq a_{ki} \). Thus, \( i \) is the best response to \( i \), and \( i \) is in Nash equilibrium with itself.

16a. Suppose that \( x \) and \( y \) are strictly dominant strategies: \( u(x, w) > u(z, w) \) for all \( w \) and all \( z \neq x \), and \( u(y, w) > u(z, w) \) for all \( w \) and all \( z \neq y \). Letting \( w = y \), it follows that \( u(x, y) > u(z, y) \) for all \( z \neq x \), and, letting \( w = x \), \( u(x, y) > u(z, x) \) for all \( z \neq y \). Therefore, \( (x, y) \) is in Nash equilibrium because each strategy is the best response to the other. Now suppose that there were another pair \( (x^*, y^*) \) in Nash equilibrium with \( x \neq x^* \). Then, \( u(x, y^*) \geq u(x^*, y^*) \) because \( x^* \) is the best response to \( y^* \). But, \( u(x, y^*) > u(x^*, y^*) \) because \( x \) is strictly dominant over \( x^* \) (and every other strategy). This contradiction means that there cannot be another Nash equilibrium pair.

16b. Only the mixed pair \( (1/3, 1/3) \) and \( (1/3, 1/3) \) are in equilibrium.

17. Collective stability simply means that a strategy is in Nash equilibrium with itself. In terms of equation 12, one must show that there is no better response to TFT than TFT. In particular, it must be the case that \( 1/(1 - \rho) \geq 2 \). This occurs only when \( \rho \geq 1/2 \). On the other hand, All-C is not collectively stable for any value of \( \rho \) because All-D cannot be the best response to itself. For every value of \( \rho \) (except \( \rho = 1 \)), \( 2/(1 - \rho) \geq 1/(1 - \rho) \)
17a. If \( u^* \) is an ESS strategy, then it is its own best response (this is part of the definition), hence in Nash-equilibrium with itself.

18. TFT earns 4 against itself and -1 against All-D. All-D earns 2 against TFT and 0 against itself. TFT, therefore, expects to earn \( 4(1-\epsilon) - \epsilon \) while All-D expects to earn \( 2(1-\epsilon) + \epsilon \). If TFT is to do better than All-D, it must be true that \( 4(1-\epsilon) - \epsilon > 2(1-\epsilon) + \epsilon \). From this inequality it follows that \( \epsilon < 2/3 \).

19. The expected earnings of TFT, from equation 15, are \( (2/3)(1/(1-0.65) - 0.33) = 1.57 \). The expected earnings of All-D are \( 2(1-\epsilon) = 1.33 \).

20. The best way is to graph \( u(\alpha, \beta) \) and \( u(\beta, \beta) \). The first is always greater than the second whenever \( \beta \) is not equal to \( 2/3 \) and the two are equal when \( \beta = 2/3 \).

21. In the Hawk-Dove game, there is only one strategy that is in Nash equilibrium with itself, \( (\rho/C, 1 - \rho/C) \), and so this is the only candidate for evolutionary stability. Call this strategy \( \alpha \). If Column plays \( \alpha \), then all strategies produce the same reward for Row: \( u(\alpha, \alpha) = u(\beta, \alpha) \) for all strategies \( \beta \). Therefore, we must show that \( \alpha \) is strictly better against \( \beta \) than \( \beta \) is itself: \( u(\alpha, \beta) > u(\beta, \beta) \) for \( \beta \neq \alpha \). Let \( \beta = (p, 1-p) \) where \( p \neq \rho/C \). This involves a bit of algebraic manipulation, but it can be done.

22. There is just one: pure defection.

23. \( x^*=(1/3,1/3,1/3) \) is in Nash-equilibrium with itself for small positive or negative \( \lambda \); if one of them is playing this strategy, any response produces the utility \(-\lambda/3\). To be an ESS strategy, if \( u(y,x^*) = u(x^*,x^*) \), then \( u(x^*,y) > u(y,y) \). Suppose that \( \lambda < 0 \) and \( y \) is any pure strategy. \( u(y,y) = -\lambda \). \( u(x^*,y) = -\lambda/3 \). Thus, \( u(x^*,y) < u(y,y) \) and \( x^* \) is not ESS. However, if \( \lambda > 0 \), then \( x^* \) is stable against any pure strategy.

25. TFT is ESS if \( r/(1-p) > s+\rho p/(1-p) \)