12th International Congress on Mathematical Education
Program Name XX-YY-zz (pp. abcde-fghij)
8 July – 15 July, 2012, COEX, Seoul, Korea (This part is for LOC use only. Please do not change this part.)

# POLYNOMIAL CALCULUS: RETHINKING THE ROLE OF CALCULUS IN HIGH SCHOOLS

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*In spite of calculus reform in the United States during the 1990's there has been limited or no lasting pedagogical innovation in the conceptual foundations of the elementary calculus for almost 200 years. That is, at all levels calculus is taught today essentially as Cauchy taught it in the 1820s. This project, Polynomial Calculus, is informed by classroom experiences and rethinking approaches for developing understanding of elementary calculus concepts for increasing access by different populations of students. For over 25 years the Algebra Project has worked to raise the level of mathematics literacy for students from underserved communities within the US. Currently, several educational professionals affiliated with the Algebra Project Network are collaboratively working on a different approach to developing understanding of concepts from the elementary calculus, referred to as the Polynomial Calculus Project.*

*calculus, conceptual understanding, experiential pedagogy*

## INTRODUCTION

Calculus has historically functioned as the entry point for advanced mathematics study between secondary and post-secondary studies. In this paper we present a reconceptualization of introductory differential calculus without the advanced techniques of limits or infinitesimals. This Polynomial Calculus offers an instructional approach that affords student access to calculus conceptual understanding with only secondary school mathematics courses, such as Algebra I, Geometry, or Algebra II. This work allows unprecedented access to calculus for students and allows for a critical distinction between alternative conceptual foundations of the elementary calculus and the traditional logical foundations of the discipline. This distinction creates new trajectories for consideration with respect to when and how the foundations of advanced mathematics study might ensue. Additionally, this work appears aligned with recent collective thinking espoused from both mathematicians and mathematics educators. In March 2012 the Mathematical Association of America (MAA) and the National Council of Teachers of Mathematics (NCTM) issued a joint position statement on the relationship between secondary and post-secondary calculus:

> Although calculus can play an important role in secondary school, the ultimate goal of the K–12 mathematics curriculum should not be to get students into and through a course in calculus by twelfth grade but to have established the mathematical foundation that will enable students to pursue whatever course of study interests them when they get to college. The college curriculum should offer students an experience that is new and engaging, broadening their understanding of the world of mathematics while strengthening their mastery of tools that they will need if they choose to pursue a mathematically intensive discipline. (MAA/NCTM, 2012)

Our intent is not to provide students with a limits-based calculus course as part of their secondary school tenure, instead provide students with a strong coherent conceptual and procedural foundation for a post-secondary study of limit-based calculus. Conceivably, students possessing conceptual calculus understanding from algebraic and geometric perspectives would be able to think more critically and creatively when they are re-introduced to ideas such as slope and area as derivative and integral, more advanced concepts of mathematical analyses. In other words, students would likely be better prepared to make sense of and use the calculus for modelling and problem solving.

### School-Based Calculus: A Rationale for Change and Perspective

At the beginning of the last century, Alfred North Whitehead (1911) began his classic text, *An Introduction to Mathematics*, with what he considered the central dilemma in the teaching of mathematics:

> The study of mathematics is apt to commence in disappointment … The reason for the failure of the science to live up to its reputation is that its fundamental ideas are not explained to the student disentangled from the technical procedure which has been invented to facilitate their exact presentation in particular instances. Accordingly, the unfortunate learner finds himself struggling to acquire a knowledge of a mass of details which are not illuminated by any general conception. (p.1)

These concerns are still with us after more than 100 years. Our group is working to develop understanding of advanced mathematics concepts for students from underserved communities within the context of their present secondary mathematics courses of study: Algebra I, Geometry, or Algebra II. The theoretical framework for our approach might be described as structural because we strive to separate mathematics concepts from procedures. Specifically, we target the mathematics teaching dilemma raised by Whitehead (1911), our goal is a reconstruction of teaching advanced mathematics concepts by separating the fundamental concepts from the "technical procedures …used to facilitate their exact presentation" (p.1).

The Polynomial Calculus group is comprised of K-18 educators from across the United States and affiliated with the Algebra Project Network. We work in secondary schools as teachers and support persons, university faculty, and compassionate mathematics educators. Our mission is to develop a calculus that affords greater access to underserved students.

## The Algebra Project History

In the late 1980s there was a movement started by Robert P. Moses to address the inequities in public mathematics education. The Algebra Project began as a middle school mathematics program that had the "philosophy that access to algebra will enable students to participate in advanced high school math and science courses, which in turn are gateways to college entrance" (Moses, Kamii, Swap, & Howard, 1989, p. 423). The goal of the Algebra Project at that time was to provide experiences that would transition students from arithmetic to algebraic thinking providing more opportunities for students to take secondary mathematics and science courses at the high school level.

The Algebra Project (AP) parallels the community organizing tradition of America's Civil Rights Movement. Bob Moses, as field secretary of the Student Non-violent Coordinating Committee in Mississippi during the 1960s, was a community organizer working with poor Black sharecropper communities in the Mississippi Delta. The mission then was to assist members of those communities to gain the right to vote and consequently more fully participate in American society (Cobb & Crombie, 2010). The Algebra Project, today, continues this mission by working with students in urban and rural underserved areas in the United States whose mathematics achievement scores fall in the lower quartile.

## POLYNOMIAL CALCULUS DEVELOPMENT PROCESS

As has been the tradition of the Algebra Project, the focus for this Polynomial Calculus project team has been to develop viable mathematics instructional approaches that increase access to underserved populations of students in ways that are both culturally sensitive and experientially based. Our team worked with students in classrooms located in two urban high schools, each located in school districts in Midwestern United States cities. Within this section two examples from these high schools are shared to justify the need for Polynomial Calculus and explain the classroom-embedded developmental nature of our work.

## The Rationale for Polynomial Calculus: A Classroom Example

The students we target are those typically found in the bottom quartile of their academic class who have persistently struggled in mathematics and often benefit from being taught using non-traditional approaches. The Algebra Project believes that re-evaluating the way that mathematics is taught will enhance students learning and we cannot continue to allow students to fall behind and/or drop out of school mathematics.

For example, students took a class excursion, recorded their experiences throughout the day, and captured documentation using notes and pictures. This event created a shared experience for all students in the class and was used for develop mathematics understanding. The common experience was mathematized by the students; pictures of places visited and order preserved from the excursion were made into a physical model that portrayed our shared experience. We used this mathematical model to study movement (addition of integers), comparison (subtraction of integers), positive and negative numbers, and inequalities. Another way mathematical experiences have been utilized for learning is that key concepts are introduced and once learned are revisited from a myriad of perspectives. For example, the idea of symmetry has been used to prove line of reflection symmetry, triangular congruence, symmetry groups of rectangles, isosceles triangles. A variety of explorations using a variety of tools, such as paper folding and structures, such as the unit circle within a coordinate plane led to more complex concepts, including symmetry over polynomials, quadratic functions, and discoveries related to trigonometric properties. After students made observations, based on symmetry, such as $\cos 40^{o} = \cos -40^{o}$ they tested their theories, collaborated to write them in generalized form, $\cos \theta = \cos -\theta$, and justified their thinking with oral arguments.

As a group of passionate mathematics educators committed to all students, we see our work as an opportunity to offer access to understanding elementary calculus concepts to students who might otherwise be denied because they flourish in different environments for learning than those traditionally made available. The Polynomial Calculus will follow the trajectory as presented in these examples and be grounded in mathematical experiences that can then be used in the pursuit of developing new understandings.

The overarching process used for developing the Polynomial Calculus instruction can be described as classroom embedded professional learning for both teachers and students each, focused on developing mathematical understanding (e.g., West & Staub, 2003; Wood, 2007). Our Polynomial Calculus team is comprised of a professional developer, two high school calculus teachers, and three university mathematics educators who work as a team of peers, each contributing as they are able and everyone working to solve issues of practice. Wood (2007) characterized this type of work as "Knowledge-of-Practice" where learning is communal and inquiry-oriented in relation to improving practice and learning. When working in classrooms our focus is on students (what they do and say), and we examine and use evidence of their learning and understanding to inform instructional moves (West & Staub, 2003).

One classroom, thus far has been the hub for Polynomial Calculus development. The professional developer's primary area of contribution is mathematics content, gained through extensive experience and graduate studies in mathematics and physics. The teachers' primary area of contribution is school-based mathematics content, pedagogy, and well established relationships with students, gained through study and teaching practice. The mathematics educators' expertise focuses on mathematics content, learning, and pedagogical challenges.

This Polynomial Calculus project started with the collaborative work of the team during the 2011-2012 school year to develop units of study within a calculus classroom. The collaborative framework followed a four step process: (a) explore background content knowledge related to the calculus concept to be taught through study and discourse; (b) plan for instruction; (c) implement the instructional unit and observe student learning and challenges; and (d) debrief the lesson and make instructional decisions for improvement and next steps. This approach is not the norm for most high school calculus teachers in the US, but was afforded in part because of the extensive teacher support offered through the Algebra Project. The calculus teacher commented that the background content study was essential for the development of the instructional units of study. The second classroom example follows and exemplifies the classroom-embedded nature of the development work.

**Polynomial Calculus Developmental Process: A Classroom Example**

The goal of this unit was for students to develop an understanding of the derivative from the experiential basis of motion and the properties of polynomials. First, students investigated the behavior of polynomial graphs by observing and making conjectures about the effects of changing coefficients of the monomials that comprise the polynomial. This investigation provided an opportunity for students to notice key features of monomial contributions on the graphs of polynomials. For example, two of these features include: (a) the height of the polynomial at the y-intercept is determined by the constant monomial, and (b) the direction of the polynomial at the y-intercept is governed by the linear monomial. These ideas reemerged as students considered and made conjectures about how to find the closest line to a parabola.

Next, motion, both uniform and non-uniform, was introduced as a grounding metaphor for analyzing graphs of lines and parabolas. Students were given a set of data about three cars that were each moving forward at a constant speed along a straight road from which, students created graphs. They were asked to determine the relative speeds of the three cars, given the following information:

> At $t_0$, the Red Car is in the lead, the Blue Car is in 2$^{nd}$ Place and the Green Car is last. At $t_1$ all of the cars are the same distance down the road. At $t_2$ the Green Car is in the lead, the Blue Car is in 2$^{nd}$, and the Red Cad is last.

From this, students easily formed various arguments for why the Green Car must be the fastest. One type of argument was based on the fact that the slope of the graph for the Green Car was greater, and therefore its speed is the greatest. Another type of argument relied on the shift of the relative position of the cars over time. For example, "The Green Car started in last and then it was in first, so it must be the fastest car." Two important concepts emerged from these arguments: (a) the slope of a line represents the speed of a uniformly moving car, and (b) when one car passes another it implies that one car must be faster than the other. These ideas were revisited after the scope of the problem was expanded to include a non-uniformly moving car. Again students created position graphs of the three cars, given the following:

> The function $q(t) = t^2$ represents the position, in meters, of Car Q compared to a starting line with respect to time, in seconds. Car A moves at a constant velocity of 0.5 meter per second. At t = 1, the position of Car A the same as that of Car Q. Car B moves at a constant velocity of 2 meters per second. At t = 1, the position of Car B is the same as that of Car Q.

From these graphs students were asked to determine which of the cars was the fastest at one second. Students determined that Car B must be the fastest and Car A must be the slowest at one second. However, they were not sure how to determine the exact speed of Car Q at one second. It was clear that the car was speeding up, but what the speed was at a particular moment was unclear.

In an effort to scaffold the students' thinking about the speed of Car Q, they were asked to consider another car, Car T. This new car, Car T, moves at a constant velocity such that at time, t = 1 second, both the speed and the position of Car T are equal to that of Car Q. They were asked to sketch a graph of the position of Car T. After some thought students were convinced that the line intersecting the graph of Car Q, q(t) at t = 1 and in the same direction as the graph of Car Q at that point represents the motion of Car T. Students were asked to elaborate on why they thought this graph represented the motion of Car T. As an example, they were given an argument that Euclid may have made years ago:

> The velocity of the Car Q at t = 1 is represented by a line that touches p(t) such that, in the space between the line and the curve, no other line can be interposed.

At this point students had a metaphorical interpretation of the closest line to a curve as the line representing uniform motion with the same instantaneous speed as the non-uniform motion associated with the curve at the point of intersection. This became our operational definition of the tangent line.

Then, the work was shifted to focus on finding equations for the closest lines to quadratic graphs. Using a graphing calculator program (developed specifically for this investigation), students were given the task of determining the slope of the closest line to a parabola initially by a trial and error method. The calculator program randomly generated a parabola, a point on the parabola and asked the student to determine the slope of the closest line to the parabola at the given point. Students entered a slope and the calculator responded by graphing the line with that slope through the point of intersection and indicating whether that line was the closest line to the graph, or if there was a line closer. The calculator displayed the last two trials graphically and symbolically – displayed the equation of both the parabola and the line for the last slope entered.

When students were unable to distinguish which of two lines was closest to the parabola, usually because of the calculator's screen resolution, students were encouraged to view the table of values for the graphs, which was also provided by the program. Students started the exercise with educated guesses but quickly moved to basing their choices of slope values on both the graphical and numerical data that were provided by the program.

Students were asked to find ten closest lines, at least three of which were at a point on the y-axis, and to consider the equations of the closest lines that passed through the y-intercept of a quadratic. They were asked to describe any patterns they noticed in the equations for those closest lines and to make conjectures about how to find the closest line to a quadratic at the y-intercept of the quadratic.

At this point, our earlier thinking and conjectures about polynomials and the influence of their component monomials on graphic behaviors became useful. Students noticed that, at the y-intercept, the graph of the line and the graph of the quadratic had the same features: both graphs had the same height (constant monomial) and direction (linear monomial). Students noticed that the closest lines at the y-intercept had the same equation as the quadratic, minus the quadratic monomial term. In other words, the equation for the closest line agrees with the equation for the quadratic to first order. In summary, this investigation afforded students an initial understanding of the relationship between the equations of a quadratic and the tangent line to the quadratic at its y-intercept.

**Discussion: Polynomial Calculus Content Analysis**

The standard approach to the differential calculus starts with the notions of the slope of the secant lines and the limit of their difference quotients. This limiting procedure yields the definition of the derivative. As part of our general concept design within the Polynomial Calculus Project we try to establish an experiential basis for the central concepts introduced in a domain of study. In the second classroom example, the experiential grounding metaphors were the notions of movement and speed. Piaget (1972) framed this epistemological perspective as follows:

> Observation shows, in fact, that there exists a basic intuition of speed, independent of any idea of duration and resulting from the primal concept of order … namely the intuition of kinematic *overtaking*. If a moving object A is behind B at instant T1 and passes in front of moving object B at an instant T2, it is judged to be faster, and this holds for all ages: nothing intervenes here except temporal order (T1 before T2) and spatial order (behind and in front of), and there is no consideration of duration or space transversed. Speed is therefore initially independent of durations. (p. 7)

This shift in perspective moves the initial task of establishing the tangent line and its slope, the derivative, from one than was computationally intense in the limit-based formulation of the calculus to one that was experientially and perceptually grounded – determining the closest line to the graph of an accelerating car.

It is worth noting that the initial exercise engages the concept of the tangent line at a level that is appropriate for a secondary Algebra I course. The limit-based definition of the tangent line as the limit of secant lines where the two points of intersection merge is a direct relationship between the tangent line and the secant lines of a curve. Moreover, the relationship between the tangent line and the curve is indirect and perhaps less accessible to novice mathematical thinkers and learners. Characterizing the tangent line as the closest line to the curve establishes a direct relationship between the tangent line and the curve which students can examine through geometric (calculator generated graphs), arithmetic (table of values), and algebraic (equations) representations of these mathematical objects. In this exercise we replaced the limiting procedure used in the definition of the tangent line with an equivalence relation. The closest line to the parabola at a given point (in this example, at the y-intercept) was defined as the line that agrees with the curve to first order. *Agreement to first order* is an equivalent relation and as such it targets for examination and reflection those features which, in this case, the parabola and tangent line have in common.

Lang (1986) defines two problems we face in the initial study of the calculus:

> One of them is to give the correct geometric idea which allows us to define the tangent line to a curve, and the other is to test whether this idea allows us to compute effectively the tangent line when the curve is given by a simple equation with numerical coefficients. . . . it is a remarkable thing that our solution of the first problem will in fact give us a solution to the second. (p. 58)

From our choice of definition the chain of reasoning is the following. The geometric condition we are interested in determining is the closest line to the parabola. This geometric condition determines an arithmetic condition. The closest line to the parabola requires the best numerical agreement in the coordinates of the parabola and the line. In turn the best numerical agreement in the coordinates requires the best agreement in the algebraic functions that generate the coordinate values. The best algebraic agreement between the closest line and the parabola, or the closest line and any polynomial for that matter, requires agreement to first order. In this reformulation the algebraic notion of equivalence replaces the analytic notion of limit as the key concept used in defining the central mathematical objects studied by the elementary calculus.

## SUMMARY

### Significance of the Polynomial Calculus

Clearly, if the Polynomial Calculus can increase access of elementary calculus conceptual understanding to underserved student populations their opportunities for future mathematics success are improved, independent of their chosen paths for future learning or work. Further, if students understand some elementary calculus concepts and are able to make sense of and use this understanding for post-secondary calculus, everyone benefits, the learner, learning institutions whose students are unable to matriculate due to mathematics and other calculus based fields of study, and our 21$^{st}$ Century technology driven society that craves more scientists and engineers.

Many different stake holders are focused on and planning for improving mathematics learning and teaching and many approaches within the United States include remediation and intervention, but our approach differs because we favor acceleration. The acceleration we endorse is accomplished through a process of affording students’ opportunities to apply current concepts and skills to the development of advanced topics. Our Polynomial Calculus is an example of the application of secondary algebra and geometry to the advanced concepts of the Calculus. In a reversal of Felix Klein’s formulation, we have introduced advanced mathematics from an elementary standpoint.

We suggest a drastic change be made to the way Calculus is taught, to whom, and by when. We wholeheartedly adopt the MAA and NCTM (2012) joint position statement that suggests secondary education focus on laying a foundation for calculus and not teach traditional calculus. This is especially germane for underserved student populations, who rarely if ever are afforded access to Calculus, elementary or otherwise. As members of the Algebra Project Network of professional educators we wholeheartedly accept our responsibility to develop mathematics literacy among underserved mathematics learners, not through remediation but by acceleration. We hope to excite and entice these young scholars to mathematical understanding and greatness through our efforts.

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