WIGNER FUNCTION

- In QM, we know that $p$ and $x$ are not compatible operators
  $=>$ we cannot measure $x$ and $p$ at the same time

- So how can we define a joint probability for (transverse) position and (transverse) momentum?

- One possible interpretation: Wigner Function $W$

$$W(\vec{r}, \vec{p}) = \frac{1}{(2\pi\hbar)^3} \iiint d^3\vec{r}' e^{i\vec{p}\cdot\vec{r}'/\hbar} \psi^*(\vec{r} - \vec{r}') \psi\left(\vec{r} + \frac{\vec{r}'}{2}\right)$$

- Can rigorously calculate things like

$$\left\langle \frac{X P_x + P_x X}{2} \right\rangle = \iiint d^3\vec{r} \iiint d^3\vec{p} x_p W(\vec{r}, \vec{p})$$

- Disadvantage: not guaranteed to be positive; oscillatory

- Limited usefulness to address our question
TECHNICAL DETAILS OF THE WIGNER FUNCTION

\[
W(\vec{r}, \vec{p}) = \frac{1}{(2\pi\hbar)^3} \int_{\text{Phase Space}} d\vec{r}' e^{-i\vec{r}' \cdot \vec{p}' \hbar} \psi^* \left( \vec{r} - \frac{\vec{r}'}{2} \right) \psi \left( \vec{r} + \frac{\vec{r}'}{2} \right)
\]

lets work on it in just 1-dim (1 spatial coordinate and 1 momentum):

\[
W(x, p) = \frac{1}{2\pi\hbar} \int dx' e^{-ipx'/\hbar} \psi^* \left( x - \frac{x'}{2} \right) \psi \left( x + \frac{x'}{2} \right).
\]

What should this function satisfy if it is to be taken as a joint probability density?

- It should be real: \( W = W^* \)

This is already satisfied, for taking the complex conjugate gives

\[
W(x, p) = \frac{1}{2\pi\hbar} \int dx' e^{ipx'/\hbar} \psi \left( x - \frac{x'}{2} \right) \psi^* \left( x + \frac{x'}{2} \right)
\]

and it is always possible to take \( x'' = -x' \), in which case we have \( W = W^* \).
Integrate now the Wigner function over all momenta
\[ \int_{-\infty}^{\infty} dpW(x, p) = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dpe^{-ipx'/\hbar} \psi \left( x - \frac{x'}{2} \right) \psi^* \left( x + \frac{x'}{2} \right) \]

Integrate now the Wigner function over all coordinates
\[ \int_{-\infty}^{\infty} dxW(x, p) = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx e^{-ipx'/\hbar} \psi \left( x - \frac{x'}{2} \right) \psi^* \left( x + \frac{x'}{2} \right) \]

this is not straightforward since \( \psi \) and \( \psi^* \) also depend on \( x \). Introduce, however, the following change of variables
\[ x_1 = x + \frac{x'}{2} \]
\[ x_2 = x - \frac{x'}{2} \]

the inverse transformation is then
\[ x' = x_1 - x_2 \]
\[ x = \frac{1}{2} (x_1 + x_2) \]

and the Jacobian for this transformation is
\[ \left| \frac{\partial (x, x')}{\partial (x_1, x_2)} \right| = \left| \begin{array}{cc} \frac{\partial x}{\partial x_1} & \frac{\partial x}{\partial x_2} \\ \frac{\partial x'}{\partial x_1} & \frac{\partial x'}{\partial x_2} \end{array} \right| = \left| \begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{array} \right| = 1 \]

thus \( dx dx' = dx_1 dx_2 \) and
\[ \int_{-\infty}^{\infty} dxW(x, p) = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 e^{-ipx_1/\hbar} e^{ipx_2/\hbar} \psi^* (x_2) \psi (x_1) \]

This is not true, since Wigner function can also be negative, but we just ignore
EXAMPLE: HARMONIC OSCILLATOR

- **Ground state:** \( \psi(x) = \left( \frac{m\omega}{\hbar \pi} \right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} \)
  - No correlation between \( x \) and \( p \) at all.
  - Both are Gaussians with minimum width
  - \( \langle x^2 \rangle \langle p^2 \rangle = \frac{1}{2} \cdot \frac{1}{2} \hbar^2 \)

- **Higher excited states** (\( \psi_n \))
  - Both \( \langle x^2 \rangle \) and \( \langle p^2 \rangle \) increase by factor \( 2n + 1 \)
  - However, \( \langle x^2 p^2 \rangle \) increases less so that correlation becomes negative:
    - For 1\(^{\text{st}}\) excited state, \( r = -0.667 \)
    - For 2\(^{\text{nd}}\) excited state, \( r = -0.857 \)
    - For 3\(^{\text{rd}}\) excited state, \( r = -0.923... \)
  - Due to negative-going parts of Wigner Function

*) Result: \( \sqrt{\frac{m\omega}{\hbar \pi}} e^{-\frac{m\omega x^2}{\hbar}} \sqrt{\frac{1}{\pi \hbar m\omega}} e^{-\frac{p^2}{\hbar m\omega}} \)
Wigner Distributions:

\[ \rho(x, \vec{k}_\perp, \vec{b}_\perp) \quad \text{3+2D map} \]
Wigner distributions

$(x, \vec{k}_\perp, \vec{b}_\perp)$

Impact parameter distributions

$(x, \vec{b}_\perp)$

TMDs

$(x, \vec{k}_\perp)$

PDFs

$(x)$

GTMDs

$(x, \vec{k}_\perp, \Delta^+ = 0, \tilde{\Delta}_\perp)$

GPDs

$(x, \Delta^+ = 0, \tilde{\Delta}_\perp)$

2D Fourier transform

$\int dx$

FormFactors

$(Q^2)$

Lorcé, BP, Vanderhaeghen, JHEP05 (2011) 041
The unpolarized GPD $H$

$H^q(x, t, b_{\perp})$ = $\int \rho \, d^2 b_{\perp} \cdot b_{\perp} \cdot b_{\perp}$


The graph shows the extrapolation of $H^q(x, t, b_{\perp})$ in the unmeasured $x$-range.
Angular Momentum Relation (Ji’s Sum Rule)

X. Ji, PRL 78 (1997) 610

quark and gluon contribution to the nucleon spin

\[ J^{q,g} = \frac{1}{2} \int_{-1}^{1} dx \ x \ (H^{q,g}(x, \xi, 0) + E^{q,g}(x, \xi, 0)) \]

\[ J^q = L^q + S^q \]

\[ J^g = L^g + S^g \]

\[ \frac{1}{2} \Delta \Sigma \text{ from DIS} \]

\[ \frac{1}{2} \Delta g \text{ from DIS} \]

- Requires extrapolation at t=0
- Requires spanning x at fixed values of \( \xi (\xi = 0 \text{ is the most convenient}) \)
- Does not have an interpretation as angular momentum density as a function of x
Lattice Calculations of Angular Momentum

\[ L^u + L^d \approx 0 \]
\[ J^u > 0, J^d \approx 0 \]

\[ L^u + L^d \approx 33\% \]
\[ (m_\pi = 330 \text{ MeV}) \]

Consistent with recent lattice calculation at the physical pion mass
C. Alexandrou et al., PRL 119 (2017) 142002

Deka et al., PRD 91 (2015) 014505
MORE ON TMDS:

- Adding the spin correlation between $x$ and $k_{\perp}$
- Correlation between $x$, $k_{\perp}$ and spin polarized quarks and/or polarized target
- Unpolarized TMD
- Correlation between $x$ and $k_{\perp}$
First hints of sign change

\[ p^\uparrow p \rightarrow W^\pm /Z \]

@ RHIC-STAR Coll.
PRL 116(2016)132301

Drell-Yan \[ \pi p \rightarrow \mu \mu X \]

@ COMPASS
PRL 119(2017)112002

\[ W^+ \rightarrow l^+ \nu \]

\[ W^- \rightarrow l^- \nu \]

sign change

no sign change

sign change

no sign change
Wigner Distributions (WD) and GTMDs from

Exclusive dijet production in ep DIS (gluon GTMDs)

Hatta, Xiao, Yuan, PRL 116 (2016) 202301
Hatta, Nakagawa, Xiao, Yuan, Zhao, PRD 95 (2017) 114032
Ji, Yuan, Zhao, PRL 118 (2017) 192004

Exclusive dijet production in pA UPC (gluon GTMDs)

Hagiwara, Hatta, Pasechnik, Tasevsky, Teryaev, PRD 96 (2016) 034009

Exclusive double quarkonia production in nucleon-nucleon collisions (gluon GTMDs)

Bhattacharya, Metz, Ojha, Tsai, Zhou, arXiv:1802.10550
Boussarie, Hatta, Xiao, Yuan, arXiv: 1807.08697

Exclusive pion-nucleon double Drell-Yan (quark GTMDs)

Bhattacharya, Metz, Zhou, PLB 771 (2017) 396