Walker-Robertson metric

\[ ds^2 = dt^2 - a^2(t) \left[ dr_c^2 + S_K(r_c) \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right) \right] \]

- Closed Universe, positive curvature \((K = +1)\):
  \( S_K(r_c) = \sin(r_c) \)

- Flat Universe, no curvature \((K = 0)\):
  \( S_K(r_c) = r_c \)

- Open Universe, negative curvature \((K = -1)\):
  \( S_K(r_c) = \sinh(r_c) \)
Important equations

Hubble law: \( v_r = \frac{dD}{dt} = \dot{a}(t) r_c = \frac{\dot{a}(t)}{a(t)} D(t) =: H(t) D \). At present:

\[ H_0 = H(t_0) = \frac{68 - 70 \text{ km/s}}{1 \text{ Mpc}} \approx \frac{1}{14 \cdot 10^9 \text{ yr}} \]

Speed of light in co-moving coordinates: \( \frac{dr_c}{dt} = \frac{c}{a(t)} \).

Redshift for light emitted at \( t \) and received at \( t_0 \):
\[ z = \frac{a(t_0)}{a(t)} - 1 \]

Invariant distance of object at time of emission:
\[ r_c(\text{em.}) = \int_{t}^{t_0} \frac{c}{a(t')} dt' \Rightarrow D(\text{em.}, t) = a(t)r_c(\text{em.}); D(\text{em.}, \text{today}) = a(t_0)r_c(\text{em.}). \]

General (Walker-Robertson) metric:
\[ ds^2 = dt^2 - a^2(t) \left[ dr_c^2 + S_K^2(r_c) \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \right]. \]

Transformation to any other local coordinate system according to Special Relativity.

Closed Universe, positive curvature \((K = +1)\):
\[ S_K(r_c) = \sin(r_c) \]

Flat Universe, no curvature \((K = 0)\):
\[ S_K(r_c) = r_c \]

Open Universe, negative curvature \((K = -1)\):
\[ S_K(r_c) = \sinh(r_c) \]
Important equations – questions:

Note: in the following slides, \( t = t_0 = 0 \) refers to "today"

1. What determines \( a(t) \)?
2. What determines \( K = -1, 0 \text{ or } 1 \)?
3. What are the initial conditions?

Escape velocity of mass \( m \) at some distance \( r = a(t) \cdot r_c \) from "center":

\[
\frac{m}{2} v^2 = \frac{GMm}{r} = \frac{G4\pi r^3 \rho m}{3r} \implies v^2 = \frac{8\pi r^2 G\rho}{3} \implies a^2 = a^2 \frac{8\pi G \rho}{3} \implies \rho = \frac{3}{8\pi G} \text{ } H^2
\]

since \( r = a \cdot r_c \) and \( v = \frac{da}{dt} r_c \)

Critical density: \( \rho_c(t) = \frac{3H^2(t)}{8\pi G} \). Calculate today's value: \( 10^{-26} \text{ kg/m}^3 = 6 \text{ protons} \)

More general:\n
\[
E = \frac{m}{2} v^2 - \frac{GMm}{r} \implies \frac{E}{T_{kin}} = 1 - \frac{8\pi G \rho r^2}{3v^2} = 1 - \frac{8\pi G \rho}{3H^2(t)} = 1 - \frac{\rho(t)}{\rho_c(t)}
\]

Einstein: \( H^2 \frac{E}{T_{kin}} = -\frac{k c^2}{a^2(t)} ; \ k = -1,0,1 \quad k = \text{sign and } a(t) = \text{radius of curvature} \)

\[
\implies H^2(t) = H^2(0) \frac{\rho(t)}{\rho_c(t)} - \frac{k c^2}{a^2(t)} = H^2(0) \left( \frac{\rho(t)}{\rho_c(0)} - \frac{k c^2}{H^2(0)a^2(t)} \right)
\]
Consequences:

Found: Critical density: \[ \rho_c(t) = \frac{3H^2(t)}{8\pi G}. \]  
\[ H^2(t) = H_0^2 \left( \frac{\rho(t)}{\rho_c(0)} - \frac{kc^2}{H_0^2a^2(t)} \right) \]

Closed Universe (positive curvature): \[ \rho_{tot} = \rho_M + \rho_R + \rho_\Lambda > \rho_c \Rightarrow K = 1, S_K(r_c) = \sin(r_c). \]
Flat Universe (no curvature): \[ \rho_{tot} = \rho_c \Rightarrow K = 0, S_K(r_c) = r_c. \]
Open Universe (negative curvature): \[ \rho_{tot} < \rho_c \Rightarrow K = -1, S_K(r_c) = \sinh(r_c). \]

Note: \( k, c, H_0 \) are constants and \( a^2(t) > 0 \Rightarrow \) once open(closed, flat), always open (c,f)

Contributions to density (r.h.s.):

1. Matter (cold, massive particles)
2. Radiation (ultrarelativistic particles)
3. Dark energy (cosmological constant)
4. Curvature (intrinsic property of universe)

\[ \Omega_M(t) = \frac{\rho_M(t)}{\rho_c(0)} \]
\[ \Omega_R(t) = \frac{\rho_R(t)}{\rho_c(0)} = \frac{\langle \epsilon_R \rangle}{\rho_c(0)c^2} \]
\[ \Omega_\Lambda(t) = \frac{\rho_\Lambda(t)}{\rho_c(0)} = \frac{\langle \epsilon_\Lambda \rangle}{\rho_c(0)c^2} \]
\[ \Omega_K(t) = -\frac{kc^2}{H_0^2a^2(t)} \]

Sum of all 4 must be equal to 1 at \( t = 0 \)
Important equations 2

Evolution Equation:

\[ H^2(t) = \frac{\dot{a}^2(t)}{a^2(t)} = \frac{8\pi G}{3} \rho_{\text{tot}}(t) - \frac{Kc^2}{a^2(t)} = H_0^2 \left( \frac{\rho_{\text{tot}}(t)}{\rho_c(t_0)} - \frac{Kc^2}{H_0^2 a^2(t)} \right) = H_0^2 \left( \Omega_M(t) + \Omega_R(t) + \Omega_\Lambda(t) + \Omega_K(t) \right) \]

\[ \Rightarrow \dot{a}(t) = a(t)H_0\sqrt{\left( \Omega_M(t) + \Omega_R(t) + \Omega_\Lambda(t) + \Omega_K(t) \right)} \Rightarrow \frac{da}{a_0} = \sqrt{\frac{a^2}{a_0^2}} \left( \Omega_M(t) + \Omega_R(t) + \Omega_\Lambda(t) + \Omega_K(t) \right) H_0 dt \]

with the following ingredients:

1) Matter (both baryonic and dark matter), non-relativistic, density due to mass:

\[ \Omega_M(t) = \frac{\rho_M(t)}{\rho_c(t_0)} = \frac{\rho_M(t_0)}{\rho_c(t_0)} \frac{a_0^3}{a^3(t)} = \Omega_M^0 \frac{a_0^3}{a^3} \]

(Note the “mixed definition” where \( \rho_c \) is always taken at today’s value). Today: \( \Omega_M^0 \approx 0.3 \), roughly 26% dark matter and 4% baryons.

2) Radiation (any relativistic particles, including photons, neutrinos in the early Universe and ultra-hot matter):

\[ \Omega_R(t) = \frac{\rho_R(t)}{\rho_c(t_0)} = \frac{\rho_R(t_0)}{\rho_c(t_0)} \frac{a_0^4}{a^4} = \Omega_R^0 \frac{a_0^4}{a^4} \]

\( \epsilon(t) = \text{energy density} \propto T^4; T(t) = T(t_0) \frac{a_0}{a(t)} \). Today: \( \Omega_R^0 = 8.24 \cdot 10^{-5} \) (mostly due to CMB)

3) Dark energy (cosmological constant \( \Lambda \)):

\[ \Omega_\Lambda(t) = \frac{\rho_\Lambda(t)}{\rho_c(t_0)} = \frac{\rho_\Lambda(t_0)}{\rho_c(t_0)} \text{ (const.)} = \Omega_\Lambda^0 \]

\( \Omega_\Lambda^0 \approx 0.7 \).

4) Curvature:

\[ \Omega_K(t) = \frac{Kc^2}{H_0^2 a^2(t)} = \Omega_K^0 \frac{a_0^2}{a^2} \]

Note: By definition \( \Omega_K^0 = 1 - \Omega_M^0 - \Omega_R^0 - \Omega_\Lambda^0 \). In principle, can by negative (open Universe) or positive (closed Universe). Today’s value unknown but very close to 0 (within 2%). Must have been extremely close to 0 in the early Universe (Inflation predicts 0).
Evolution – several cases

\[
\frac{da}{a} = \left( \Omega_M^0 \frac{a_0^3}{a^3(t)} + \Omega_R^0 \frac{a_0^4}{a^4(t)} + \Omega_\Lambda^0 - \frac{Kc^2}{H_0^2 a^2(t)} \right)^{1/2} H_0 dt \Rightarrow \frac{da}{\left( \Omega_M^0 \frac{a_0}{a(t)} + \Omega_R^0 \frac{a_0^2}{a^2(t)} + \frac{a^2(t)}{a_0^2 \Omega_\Lambda^0} - \frac{Kc^2}{a_0^2 H_0^2} \right)^{1/2} = a_0 H_0 dt}
\]

Matter dominance:

\[ a^{1/2} \, da = a_0^{3/2} \sqrt{\Omega_M^0} H_0 dt \Rightarrow \frac{2}{3} \left( a^{3/2} (t) - a_0^{3/2} \right) = a_0^{3/2} \sqrt{\Omega_M^0} H_0 t \Rightarrow \frac{a(t)}{a_0} = \left( 1 + \frac{3}{2} \sqrt{\Omega_M^0} H_0 t \right)^{2/3} \]

Radiation dominance:

\[ a \, da = a_0^2 \sqrt{\Omega_R^0} H_0 dt \Rightarrow \frac{1}{2} \left( a^2 (t) - a_0^2 \right) = a_0^2 \sqrt{\Omega_R^0} H_0 t \Rightarrow \frac{a(t)}{a_0} = \left( 1 + 2 \sqrt{\Omega_R^0} H_0 t \right)^{1/2} \]

Dark energy dominance:

\[ \frac{da}{a} = \sqrt{\Omega_\Lambda^0} H_0 dt \Rightarrow \ln \left( \frac{a(t)}{a_0} \right) = \sqrt{\Omega_\Lambda^0} H_0 t \Rightarrow \frac{a(t)}{a_0} = e^{\sqrt{\Omega_\Lambda^0} H_0 t} \]

Curvature dominance (only possible if \( K = -1 \)):

\[ da = c dt \Rightarrow a(t) = a_0 + ct \]
The age and ultimate fate of the universe can be determined by measuring the Hubble constant today and extrapolating with the observed value of the deceleration parameter, uniquely characterized by values of density parameters ($\Omega_M$ for matter and $\Omega_\Lambda$ for dark energy). A "closed universe" with $\Omega_M > 1$ and $\Omega_\Lambda = 0$ comes to an end in a Big Crunch and is considerably younger than its Hubble age. An "open universe" with $\Omega_M \leq 1$ and $\Omega_\Lambda = 0$ expands forever and has an age that is closer to its Hubble age. For the accelerating universe with nonzero $\Omega_\Lambda$ that we inhabit, the age of the universe is coincidentally very close to the Hubble age.

An observation stemming from this theorem is that seeing objects recede from us on Earth is not an indication that Earth is near to a center from which the expansion is occurring, but rather that every observer in an expanding universe will see objects receding from them.

**Ultimate fate and age of the universe**

The value of the Hubble parameter changes over time, either increasing or decreasing depending on the sign of the so-called deceleration parameter, which is defined by:

$$a(t) = \frac{1}{H},$$

where $H$ is the Hubble parameter and $a(t)$ is the scale factor.

In a universe with a deceleration parameter equal to zero, it follows that $H = 1/t$, where $t$ is the time since the Big Bang. A non-zero, time-dependent value of simply requires integration of the Friedmann equations backwards from the present time to the time when the comoving horizon size was zero.

It was long thought that $q$ was positive, indicating that the expansion is slowing down due to gravitational attraction. This would imply an age of the Universe less than $1/H$, which is about 14 billion years. For instance, a value for $q$ of 1/2 (once favoured by most theorists) would give the age of the Universe as $2/(3H)$. The discovery in 1998 that $q$ is apparently negative means that the Universe could actually be older than $1/H$. However, estimates of the age of the universe are very close to $1/H$.

**Olbers' paradox**

The expansion of space summarized by the Big Bang interpretation of Hubble's Law is relevant to the old conundrum known as Olbers' paradox: if the Universe were infinite, static, and filled with a uniform distribution of stars, then every line of sight in the sky would end on a star, and the sky would be as bright as the surface of a star. However, the night sky is largely dark. Since the 17th century, astronomers and other thinkers have proposed many possible ways to resolve this paradox, but the currently accepted resolution depends in part on the density parameters and the nature of dark energy.
Consequences for our Universe

All Omegas as Fractions of Total

Distances from emitter at time $t$ observed today