Classical Mechanics - Midterm Exam - Solution

Problem 1)

We choose as generalized coordinates the position $x$ of the mass $M$ relative to the position where the spring is relaxed, the position $y$ of the c.o.m. of the cylinder relative to the same position, and the angle $\theta$ that an arbitrary radial scribe mark on the face of the cylinder makes with the “down” direction (assumed to be zero if both $x$ and $y$ are zero). The constraint of rolling without slipping can be expressed through $g(x,y,\theta) = y - x + R \theta = 0$.

The kinetic energy of the block is just $\frac{M}{2} x^2$ and the kinetic energy of the cylinder can be written as the sum of the kinetic energy of the center of mass motion, $\frac{m}{2} y^2$ and the rotation around the center of mass, $\frac{I}{2} \dot{\theta}^2 = \frac{mR^2}{4} \dot{\theta}^2$. The full Lagrangian with constraint can then be written as $L = \frac{M}{2} \dot{x}^2 + \frac{m}{2} \dot{y}^2 + \frac{mR^2}{4} \dot{\theta}^2 - \frac{k}{2} x^2 - \lambda g$ and the Euler-Lagrange Equations of Motion become:

$$M \ddot{x} = -kx - \lambda \frac{\partial g}{\partial x} = -kx + \lambda$$

$$m \ddot{y} = -\lambda \frac{\partial g}{\partial y} = -\lambda$$

$$\frac{mR^2}{2} \ddot{\theta} = -\lambda \frac{\partial g}{\partial \theta} = -R\lambda$$

We can use the middle equation to replace $\lambda$ in the first and third equation, and following the constraint, we can replace $\ddot{y}$ with

$$\lambda = -m \ddot{y} = -m(\ddot{x} - R \ddot{\theta}) \Rightarrow M \ddot{x} = -kx - m(\ddot{x} - R \ddot{\theta}) \text{ and } \frac{mR^2}{2} \ddot{\theta} = Rm(\ddot{x} - R \ddot{\theta})$$

Solving the last equation for $\ddot{\theta}$ gives $\ddot{\theta} = \frac{Rm \ddot{x}}{\frac{2}{3} mR^2} = \frac{2}{3R} \ddot{x}$ and plugging back into the first one yields $(M + m) \ddot{x} = -kx + \frac{2m}{3} \ddot{x} = \left( M + \frac{1}{3} m \right) \ddot{x} = -kx$. The result is a harmonic oscillation in $x$ with angular frequency $\omega = \sqrt{\frac{k}{M + \frac{1}{3} m}}$. If we assume that initially we displace both
the block and the cylinder on top of it by some distance $x_0$ from the spring equilibrium while keeping the initial orientation of the cylinder ($\theta(t=0) = 0$), the motion of the block is simply $x(t) = x_0 \cos \omega t$. Plugging in yields $\ddot{x} = -\frac{2\omega^2 x_0}{3R} \cos \omega t$ which, together with the initial conditions, solves $\theta(t) = \frac{2x_0}{3R} (\cos \omega t - 1)$. Finally, plugging this into the equation of constraint yields $y = x - R\theta = \frac{1}{3} x_0 \cos \omega t + \frac{2}{3} x_0$. Hence, the block makes harmonic oscillations around the equilibrium position, while the cylinder rolls such on top if that the c.o.m. of the cylinder moves only with $1/3$ of the amplitude, between the initial position $x_0$ and $x_0/3$.

Finally, we can solve $\lambda = -m\ddot{y} = -\frac{m\omega^2}{3} x_0 \cos \omega t = -\frac{mk}{3M + m} x_0 \cos \omega t$. The center of mass motion of the cylinder must be entirely due to the frictional force acting on the contact point between it and the block, so that this expression is equal to that frictional force. The normal force is equal to $mg$, so that the maximum frictional force is $\frac{mk}{3M + m} x_0 < \mu mg \Rightarrow x_0 < \frac{\mu g}{k} (3M + m)$.

**Problem 2)**

We begin by calculating the inertia tensor for the coin around its center of mass. Clearly, the symmetry axes are the rotational axis and any two other perpendicular axes, so the tensor is diagonal with the values

$$I_3 = \iiint_{\text{coin}} dr \, d\theta \, d\phi (r^2 \cos^2 \phi + r^2 \sin^2 \phi) = 2\pi \int_0^R r^3 \, dr = \frac{2\pi R^4}{4} = \frac{1}{2} MR^2$$

(t is the thickness of the coin); and

$$I_1 = I_2 = \iiint_{\text{coin}} dr \, d\theta \, d\phi (r^2 \cos^2 \phi + z^2) = \pi \int_0^R r^3 \, dr = \frac{\pi R^4}{4} = \frac{1}{4} MR^2$$

(the term with $z^2$ can be ignored since the coin is so thin).

In the 1-2-3 coordinate system given in the problem, the center of mass is displaced by a vector $(0, R, 0)$ from the origin, which according to the parallel axis theorem means we have to add the matrix $M (R^2 \mathbf{1} - R \cdot \hat{R})$ to the inertia tensor. All off-diagonal elements are
still zero, as well as the second (middle) diagonal element, so that the new moments of inertia are

$$I_1' = I_1 + MR^2 = \frac{5}{4} MR^2; \quad I_2' = I_2 = \frac{1}{4} MR^2; \quad I_3' = I_3 + MR^2 = \frac{3}{2} MR^2.$$  

Within the coordinate system given by the problem, the components of the angular velocity vector are

$$\omega_1 = \dot{\theta}; \quad \omega_2 = \sin \theta \dot{\phi}; \quad \omega_3 = \cos \theta \dot{\psi} + \dot{\psi}.$$  

(This follows either from Goldstein’s equation (4.87), p. 174, with \(y = 0\) (since the line of nodes is unrotated) or by direct inspection (checking around which axis each change of the three Euler angles will rotate the coin). We can therefore write down the kinetic energy as

$$T = \frac{1}{2} \left[ I_1 \dot{\theta}^2 + I_2 \dot{\phi}^2 + I_3 \dot{\psi}^2 \right] = 2 \left[ \frac{5}{4} MR^2 \dot{\theta}^2 + \frac{1}{4} MR^2 \sin^2 \theta \dot{\phi}^2 + \frac{3}{2} MR^2 \left( \cos \theta \dot{\psi} + \dot{\psi} \right)^2 \right].$$

Finally, the total potential energy of the system is simply

$$V = MgR \sin \theta$$

so that the full Lagrangian is

$$\mathcal{L} = \frac{1}{8} MR^2 \left( \dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 \right) + \frac{1}{4} MR^2 \left( \dot{\psi} + \cos \theta \dot{\phi} \right)^2 + \frac{M}{2} R^2 \left( \left( \dot{\psi} + \cos \theta \dot{\phi} \right)^2 + \dot{\psi}^2 \right) - MgR \sin \theta$$

$$\quad = \frac{5}{8} MR^2 \dot{\theta}^2 + \frac{1}{8} MR^2 \sin^2 \theta \dot{\phi}^2 + \frac{3}{4} MR^2 \left( \dot{\psi} + \cos \theta \dot{\phi} \right)^2 - MgR \sin \theta$$

Obviously, \(\psi\) and \(\phi\) are cyclic so the corresponding momenta are conserved. Here are all 3 generalized momenta:

$$p_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \frac{3}{2} MR^2 \left( \dot{\psi} + \cos \theta \dot{\phi} \right) = \frac{3}{2} MR^2 \omega_3 = \text{const.}$$

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{1}{4} MR^2 \sin^2 \theta \dot{\phi} + \frac{3}{2} MR^2 \left( \dot{\psi} + \cos \theta \dot{\phi} \right) \cos \theta = \frac{1}{4} MR^2 \sin^2 \theta \dot{\phi} + p_\psi \cos \theta = \text{const}$$

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{5}{4} MR^2 \dot{\theta}$$

Since all constraints are time-independent, the kinetic energy is of the usual quadratic form, so the energy function is simply the total energy (and it is conserved):
\[ E = \frac{5}{8} MR^2 \dot{\theta}^2 + \frac{1}{8} MR^2 \sin^2 \theta \dot{\phi}^2 + \frac{3}{4} MR^2 \left( \dot{\psi} + \cos \theta \dot{\phi} \right)^2 + MgR \sin \theta \]

\[
= \frac{5}{8} MR^2 \dot{\theta}^2 + \frac{1}{8} MR^2 \sin^2 \theta \left( \frac{p_v - p_v \cos \theta}{\frac{1}{4} MR^2 \sin^2 \theta} \right)^2 + \frac{3}{4} MR^2 \left( \frac{p_v}{\frac{3}{2} MR^2} \right)^2 + MgR \sin \theta
\]

\[
= \frac{5}{8} MR^2 \dot{\theta}^2 + 2 \left( \frac{p_v - p_v \cos \theta}{MR^2 \sin^2 \theta} \right)^2 \frac{p_v^2}{3MR^2} + MgR \sin \theta = \frac{1}{2} \left( \frac{5}{4} MR^2 \right) \dot{\theta}^2 + V'(\theta)
\]

To find stationary solutions with constant \( \theta \), we have to find the zeros of the first derivative of the “equivalent potential” \( V'(\theta) \):

\[
\frac{\partial V'}{\partial \theta} = 4 \left( \frac{p_v - p_v \cos \theta}{MR^2 \sin^2 \theta} \right) p_v \sin \theta - 4 \left( \frac{p_v - p_v \cos \theta}{MR^2 \sin^2 \theta} \right) \cos \theta + MgR \cos \theta =
\]

\[
4 \left( \frac{p_v - p_v \cos \theta}{MR^2 \sin^3 \theta} \right) \left( p_v \sin^2 \theta - p_v \cos \theta + p_v^2 \cos \theta \right) + MgR \cos \theta = 4 \left( \frac{p_v - p_v \cos \theta}{MR^2 \sin^3 \theta} \right) \left( \frac{p_v - p_v \cos \theta}{MR^2 \sin^3 \theta} \right) + MgR \cos \theta
\]

\[
\dot{\phi} \left( \frac{p_v - p_v \cos \theta}{\sin \theta} \right) + MgR \cos \theta = \dot{\phi} \left( p_v - \frac{1}{4} MR^2 \cos \theta \dot{\phi} \right) \sin \theta + MgR \cos \theta = 0
\]

(In the last line, we have made use of the fact that constant \( \theta \) implies constant \( \dot{\phi} \) and \( \dot{\psi} \)).

For the situation described in the problem, we have \( \dot{\psi} = v / R \) and \( \dot{\phi} = -v / a \) for a clockwise motion of the coin around the z-axis (note the relative sign!). For a given angle \( \theta \)

this yields \( p_v = \frac{3}{2} MR^2 \left( \dot{\psi} + \cos \theta \dot{\phi} \right) = \frac{3}{2} MR \left( 1 - \cos \theta R / a \right) v \) and therefore

\[
\dot{\phi} \left( p_v - \frac{1}{4} MR^2 \cos \theta \dot{\phi} \right) \sin \theta + MgR \cos \theta = -\frac{v}{a} \left( \frac{3}{2} MR \left( 1 - \cos \theta R / a \right) v + \frac{1}{4} MR^2 \cos \theta \frac{v}{a} \right) \sin \theta + MgR \cos \theta =
\]

\[
-\frac{v}{a} \left( \frac{3}{2} MRv - \frac{5}{4} MR^2 \cos \theta \frac{v}{a} \right) \sin \theta + MgR \cos \theta = -MR \left( \frac{v}{a} \right)^2 \left( \frac{3}{2} a - \frac{5}{4} R \cos \theta \right) \sin \theta + MgR \cos \theta = 0
\]

\[
\Rightarrow \left( \frac{v}{a} \right)^2 = \frac{g \cot \theta}{\frac{3}{2} a - \frac{5}{4} R \cos \theta} \Rightarrow \quad v = a \sqrt{\frac{g \cot \theta}{\frac{3}{2} a - \frac{5}{4} R \cos \theta}}
\]

These equations can be solved for \( v \), \( a \), or (in principle) \( \cos \theta \) if the other variables are given.
Problem 3)

We begin by writing down the “energy equation” for the motion:

\[ E = \frac{\mu}{2} r^2(t) + \frac{P_\phi^2}{2 \mu \sigma^2(t)} + k r^\alpha(t) = \text{const.} \]

We know that by assumption \( r(t) \) is a solution to this equation for suitable values of \( E \) and \( P_\phi \). We now want to prove that the function \( r'(t) = \lambda r(\lambda^\alpha t) \) is also a solution (the angles \( \theta \) and \( \phi \) are not affected by the “scaling” of \( r \) with \( \lambda \)). Plugging in \( r' \) instead of \( r \) yields

\[ E' = \frac{\mu}{2} r'^2(t) + \frac{P_\phi'^2}{2 \mu \sigma'^2(t)} + k r'^\alpha(t) = \frac{\mu}{2} \left( \lambda \lambda^\alpha r(\lambda^\alpha t) \right)^2 + \frac{P_\phi'^2}{2 \mu \lambda^2 r^2(\lambda^\alpha t)} + k \lambda^\alpha r^\alpha(\lambda^\alpha t) \]

\[ = \lambda^\alpha \left( \frac{\mu}{2} \lambda^{2\alpha+\sigma^2-\alpha} r^2(\lambda^\alpha t) + \frac{P_\phi'^2}{2 \mu \lambda^2 r^2(\lambda^\alpha t)} + k r^\alpha(\lambda^\alpha t) \right) \]

If we choose \( \sigma = \frac{\alpha}{2} - 1 \) and \( P_\phi' = \lambda^{\frac{\alpha}{2}} P_\phi \), then the expression inside the brackets has the exact same form as \( E(t') = E = \text{const.} \) with \( t' = \lambda^\alpha t \). We can conclude that \( E' = \lambda^\alpha E \) is indeed constant under these assumptions and therefore \( r'(t) \) is a viable solution for the equations of motion.

For the case of the harmonic oscillator, \( \alpha = 2 \) and hence \( \sigma = 0 \). This simply shows that the time-dependence of the motion is entirely independent of the amplitude, meaning for any solution \( r(t) \) any multiple of that solution is also an allowed motion.

For the case of Kepler motion, \( \alpha = -1 \) and hence \( \sigma = -3/2 \). This means that if we increase an allowed orbit uniformly by a factor \( \lambda \), we still get an allowed orbit but the motion will be slowed down by a factor \( 1 / \lambda^{3/2} \). This of course means in particular that the overall time needed for one full revolution goes up by \( \lambda^{3/2} \), meaning that the square of the period will increase proportional to the cube of \( \lambda \), which of course is in agreement with Kepler’s third law (since in particular the major half-axis will of course also increase by the same factor).