Classical Mechanics - Midterm Exam - Solution

Problem 1)

Note: I took this problem from a recent Qualifier Exam without checking the solution first. As it turns out, the formulation is a bit misleading – as some of you may have noticed, there is really no “small oscillation around the stationary/equilibrium position”, but instead a time-dependent oscillation super-imposed on an accelerated linear motion of the two masses. Therefore, the last step in the solution below is not required for full credit (in fact, an earlier version of this solution had a mistake in it!)

Let L be the “extra length” of string, i.e. the distance by which the left mass hangs below its pulley if \( r = 0 \). Then the position of the left mass is \( y_{\text{left}} = r - L \) and \( \dot{y}_{\text{left}} = \dot{r} \), while the vertical position of the right mass is \( y_{\text{right}} = -rcos\alpha \). The kinetic energy of the left mass is simply \( \frac{1}{2} m \dot{r}^2 \) and that of the right mass is the usual kinematic energy in polar coordinates, \( \frac{1}{2} (\dot{r}^2 + r^2 \dot{\alpha}^2) \). Since \( V_{\text{pot}} = mg y \) for both masses, we can write down the Lagrangian as

\[
L = \frac{m}{2} \left( 2\dot{r}^2 + r^2 \dot{\alpha}^2 \right) - mg(r - L - r \cos \alpha)
\]

The Euler-Lagrange Equations of Motion become:

1) \( \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = 2m\ddot{r} = \frac{\partial L}{\partial r} = mr \ddot{\alpha}^2 - mg(1 - \cos \alpha) \)

2) \( \frac{d}{dt} \frac{\partial L}{\partial \dot{\alpha}} = \frac{d}{dt} mr^2 \ddot{\alpha} = 2mr \dddot{\alpha} + mr^2 \ddot{\alpha} = \frac{\partial L}{\partial \alpha} = -mgr \sin \alpha \)

Clearly, if \( r = r_0 \) and \( \alpha = 0 \) (both constant), these equations are fulfilled, and we have an (indifferent) equilibrium. Now we assume that we have small deviations from this equilibrium; in particular, \( r = r_0 + \delta r \) and \( \alpha \) is small enough so that we can approximate \( \sin \alpha = \alpha \) and \( 1 - \cos \alpha = \alpha^2 / 2 \). Finally, following the hint, we can drop the first term in the 2nd row of equations, and therefore get the simple harmonic oscillator equation for \( \alpha \):

\[
\ddot{\alpha} = -\frac{g}{r} \alpha \rightarrow \alpha(t) = \alpha_0 \cos \omega t \text{ with } \omega = \sqrt{\frac{g}{r}}.
\]

Plugging into the first equation, we get

\[
\delta \ddot{r} = \frac{1}{2} r \dddot{\alpha}^2 - \frac{1}{4} g \alpha^2 = \frac{1}{2} r \alpha_0^2 \omega^2 \sin^2 \omega t - \frac{1}{4} g \alpha_0^2 \cos^2 \omega t = -g \alpha_0^2 \left( \frac{1}{4} \cos^2 \omega t - \frac{1}{2} \sin^2 \omega t \right),
\]

where we replaced \( r \omega^2 \) with \( g \). The expression in the parentheses can be written as
\[
\left( \frac{3}{8} \cos^2 \omega t - \frac{3}{8} \sin^2 \omega t - \frac{1}{8} \right) = \left( \frac{3}{8} \cos 2\omega t - \frac{1}{8} \right)
\]

This can be integrated directly to yield \( \delta r = -\frac{1}{8} g \alpha_0^2 \left( \frac{3}{2\omega} \sin 2\omega t - t \right) \), assuming that initially (in equilibrium), \( \delta r = 0 \). We can now also check that the neglected term in the second equation is of higher order:

\[
2mr \ddot{r} = 2mr \left( -\frac{1}{8} g \alpha_0^2 \left( \frac{3}{2\omega} \sin 2\omega t - t \right) \right) = \frac{1}{4} m g r \alpha_0^3 \left( \frac{3}{2} \sin 2\omega t - \omega t \right) \sin \omega t
\]

i.e., it is in 3rd order of the small parameter \( \alpha_0 \) while the other 2 terms in the 2nd equation are of order \( \alpha_0 \). Of course, this is only true initially – over time, the term linear in \( t \) grows beyond all bounds, so that this approximation becomes worse over time.

Finally, we can integrate one more time to get

\[
\delta r(t) = \frac{1}{8} g \alpha_0^2 \left( \frac{3}{4\omega^2} \cos 2\omega t + \frac{t^2}{2} \right) = \frac{3}{32} r \alpha_0^2 \cos 2\omega t + \frac{1}{8} g \alpha_0^2 \frac{t^2}{2},
\]

i.e. the masses bob up and down with twice the frequency of the pendulum on the right side, superimposed on an accelerated motion (right mass down, left mass up) with acceleration \( \frac{1}{8} g \alpha_0^2 \).

**Problem 2)**

Since the disc can only rotate around a single (z-)axis, all we need to write down the Lagrangian is the moment of inertia, \( I_3 \), around that axis, the mass \( M \) of the disc, and the distance \( D \) between the suspension point and the center of mass of the disc:

\[
L = \frac{1}{2} I_3 \dot{\phi}^2 + MgD \cos \phi
\]

The single Euler-Lagrange Equation for the system becomes

\[
in_3 \ddot{\phi} = -MgD \sin \phi \approx -MgD \phi
\]

which has the usual harmonic oscillator solution \( \phi(t) = A \cos \omega t \) with \( \omega = \sqrt{\frac{MgD}{I_3}} \).

We first observe that, given constant density \( \rho \) of the disc, the mass would be

\[
M = \pi R^2 \rho T - \pi r^2 \rho T = \pi \rho T (R^2 - r^2)
\]

where \( T \) is the thickness of the disc.

Secondly, we can calculate the center of mass of the disc (in its rest position; of course the answer doesn’t depend on that) as

\[
MD = \rho T \left( \int_0^{2R} dy \sqrt{R^2 - (y - R)^2} - \int_{R+h-r}^{R+h+r} dy \sqrt{r^2 - (y - R - h)^2} \right)
\]
where the second term subtracts the hole. For both integrals, we can substitute the integration variable:

\[
MD = qT \left( \int_{y=-R}^{y=R} dy' (y' + R) \sqrt{R^2 - (y')^2} - \int_{y=-r}^{y=r} dy' (y' + R + h) \sqrt{r^2 - (y')^2} \right)
\]

The first term in each integral integrates out to zero (odd function of \(y')\), while the second part allows us to pull out a constant and the rest is just the surface of the (large or small) disc. The final answer is

\[
D = \frac{qT (R \cdot \pi R^2 - [R + h] \pi r^2)}{M} = \frac{(MR - qT h \pi r^2)}{M} = R - h \frac{m}{M}
\]

where we define \(m = \pi qTr^2 = M \frac{r^2}{R^2 - r^2}\) as the “missing mass” of the cut-out hole.

Finally, for the moment of inertia, we use the parallel axis theorem to first calculate the moment of inertia of the big disc without the hole, and the moment of inertia of the little disc that is cut out from the big disc. The total moment of inertia is then just the difference.

For the large disc without hole, the moment of inertia around its own center would be \(\pi qTR^4/2\) according to the hint, and around O it would be \(3\pi qTR^4/2\) according to the parallel axis theorem. (The – hypothetical – mass of the disc would be just \(\pi qTR^2\)). For the small disc the results are \(\pi qTr^4/2\) and \(\frac{\pi qTr^4}{2} + (R + h)^2 \pi qTr^2\). Subtracting the two terms yields

\[
I_3 = \pi qT \left( \frac{3}{2}R^4 - \frac{1}{2}r^4 - (R + h)^2 r^2 \right)
\]

This can be algebraically transformed into

\[
I_3 = M \left( \frac{\frac{1}{2}[R^4 - r^4] + R^4 - R^2 r^2 - (2Rh + h^2)r^2}{R^2 - r^2} \right)
\]

\[
= M \left( \frac{R^2 + r^2}{2} + R^2 \right) - m(2Rh + h^2)
\]

using our previous definition. Plugging it all in yields

\[
\omega = \sqrt{\frac{MgD}{I_3}} = \sqrt{\frac{g \left( R - h \frac{m}{M} \right)}{\frac{3R^2 + r^2}{2} - (2Rh + h^2) \frac{m}{M}}} = \sqrt{\frac{g \frac{R^3 - (R + h)r^2}{2R^4 - 1 \frac{r^4}{2} - (R + h)^2 r^2}}}
\]

**Problem 3)**

We begin by writing down the “energy equation” for the motion:
We know that by assumption \( r(t) \) is a solution to this equation for suitable values of \( E \) and \( \Psi \). We now want to prove that the function \( r'(t) = \lambda r(\lambda^\alpha t) \) is also a solution, albeit perhaps with different values for energy and angular momentum (the angles \( \theta \) and \( \phi \) are not affected by the “scaling” of \( r \) with \( \lambda \)). Plugging in \( r' \) instead of \( r \) yields

\[
E' = \frac{\mu}{2} r'^2(t) + \frac{P^2}{2\mu r'^2(t)} + kr'^\alpha(t) = \text{const}
\]

If we choose \( \sigma = \frac{\alpha}{2} - 1 \) and \( P' = \lambda^{1/2} P \), then the expression inside the brackets has the exact same form as \( E(t') = E = \text{const} \) with \( t' = \lambda^\alpha t \). We can conclude that \( E' = \lambda^\alpha E \) is indeed constant under these assumptions and therefore \( r'(t) \) is a viable solution for the equations of motion.

For the case of the harmonic oscillator, \( \alpha = 2 \) and hence \( \sigma = 0 \). This simply shows that the time-dependence of the motion is entirely independent of the amplitude, meaning for any solution \( r(t) \) any multiple of that solution is also an allowed motion.

For the case of Kepler motion, \( \alpha = -1 \) and hence \( \sigma = -3/2 \). This means that if we increase an allowed orbit uniformly by a factor \( \lambda \), we still get an allowed orbit but the motion will be slowed down by a factor \( 1 / \lambda^{3/2} \). This of course means in particular that the overall time needed for one full revolution goes up by \( \lambda^{3/2} \), meaning that the square of the period will increase proportional to the cube of \( \lambda \), which of course is in agreement with Kepler’s third law (since in particular the major half-axis will of course also increase by the same factor).