Central Force Problem

Consider two bodies of masses, say earth and moon, m_E and m_M moving under the influence of mutual gravitational force of potential V(r). Now Langangian of the system is

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 sin^2\theta\dot{\varphi}^2) - V(\mathbf{r}) \tag{1}$$

where, $\mu = \frac{\mathrm{m_E.m_M}}{\mathrm{M}}$ and $\mathrm{M} = \mathrm{m_E} + \mathrm{M_m}$
Now, the generalized momenta are

$$\begin{aligned} P_r &= \frac{\partial L}{\partial \dot{r}} = \mu\dot{r} \\ P_\theta &= \frac{\partial L}{\partial \dot{\theta}} = \mu r^2\dot{\theta} \\ P_\varphi &= \frac{\partial L}{\partial \dot{\varphi}} = \mu r^2 sin^2\theta\dot{\varphi} \end{aligned} \tag{2}$$

Select the case:

$$\begin{aligned} \theta(t=0) &= 90^\circ \\ \dot{\theta}(t=0) &= 0 \end{aligned} \tag{3}$$

(always possible by orientation of the x, y, z coordinate system). The Euler-Lagrange Equation for θ is

$$\begin{aligned} \frac{dP_\theta}{dt} &= 2\mu r\dot{r}\dot{\theta} + \mu r^2\ddot{\theta} = \\ \frac{\partial L}{\partial \theta} &= \mu r^2\sin(\theta)\cos(\theta)\dot{\phi}^2 \end{aligned} \tag{4}$$

Since all other terms are zero due to our choice, it must be true that also

$$\ddot{\theta}(t=0) = 0$$

This can be expanded for higher derivatives, ultimately showing that θ must be constant at 90 degrees. This is of course due to the fact that both the magnitude and the direction of the angular momentum vector $\mathbf{L}$ is conserved, and the radius vector is always perpendicular to it. So if we choose the z-direction in the direction of $\mathbf{L}$, the equations of motion for $r(t)$ and $\varphi(t)$, are restricted to the x-y plane. We have now reduced our analysis to that of a system with 2 degrees of freedom, namely (r, φ).

From now on, we assume that the force is pointing along the direction of the relative position $\mathbf{r}$ between the two objects. We can say that for such a central force the potential depends only on the distance $|\mathbf{r}|$ between the two objects.

$$V(\mathbf{r}) = V(r) = -\frac{GM\mu}{r} \tag{5}$$

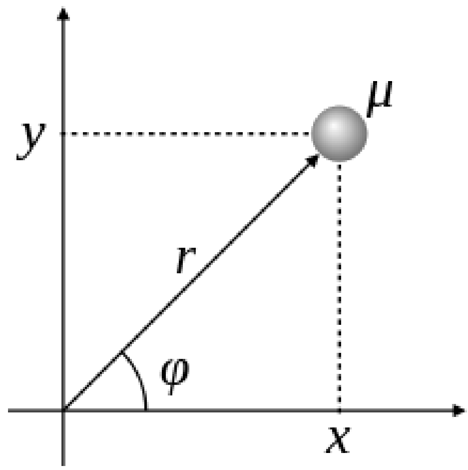

Figure 1: Motion of two body system of reduced mass μ in the central force field

Then the Langrangian for this system

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) - V(r) \tag{6}$$

From Euler-Langrange equation (ELE)

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

We get

$$\dot{P}_\varphi = 0 = \mu r^2 \ddot{\varphi} \tag{7}$$

Then, the angular momentum l is constant

$$P_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \mu r^2 \dot{\varphi} = l = \text{constant} \tag{8}$$

Similarly, from ELE for r

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = 0$$

$$\mu\ddot{r} - \mu r\dot{\varphi}^2 + \frac{\partial V(r)}{\partial r} = 0 \tag{9}$$

Then

$$\dot{P}_r = \frac{\partial \mathcal{L}}{\partial r}$$

$$\mu\ddot{r} = \mu r\dot{\varphi}^2 - \frac{GM\mu}{r^2}$$

$$\mu\ddot{r} = -\frac{\partial V(r)}{\partial r} + \frac{P_\varphi}{\mu r^3} \tag{10}$$

let a function h given by

$$h = \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) + V(r)$$

$$h = \frac{\mu}{2}\dot{r}^2 + \frac{{P_\varphi}^2}{2\mu r^2} + V(r) = E \tag{11}$$

$$h = \frac{\mu}{2}\dot{r}^2 + \frac{l^2}{2\mu r^2} + V(r) = E$$

$$\therefore h = \frac{\mu}{2}\dot{r}^2 + \text{"}V(r)\text{"} = E \tag{12}$$

where $\text{"}V(r)\text{"} = \frac{l^2}{2\mu r^2} + V(r)$ is pseudo-potential and E is the total energy. From Eq. (10),

$$\dot{r} = \pm\sqrt{\frac{2}{\mu}}\sqrt{E - \frac{P_\varphi^2}{2\mu r^2} - V(r)}$$

where the sign $\pm$ depends on $r(t)$ is increasing or decreasing at time t. It doesn't alter the trajectory.Taking '+' sign, we get

$$\int_{r_1}^{r_2} \frac{dr}{\sqrt{\frac{2}{\mu}}\sqrt{E - \frac{P_\varphi^2}{2\mu r^2} - V(r)}} = \int_{t_1}^{t_2} dt = t_2 - t_1 \tag{13}$$

Kepler's Second Law

From Eq. (7)

$$\dot{\varphi} = \frac{d\varphi}{dt} = \frac{l}{\mu r^2}$$

The differential area swept out in time dt is

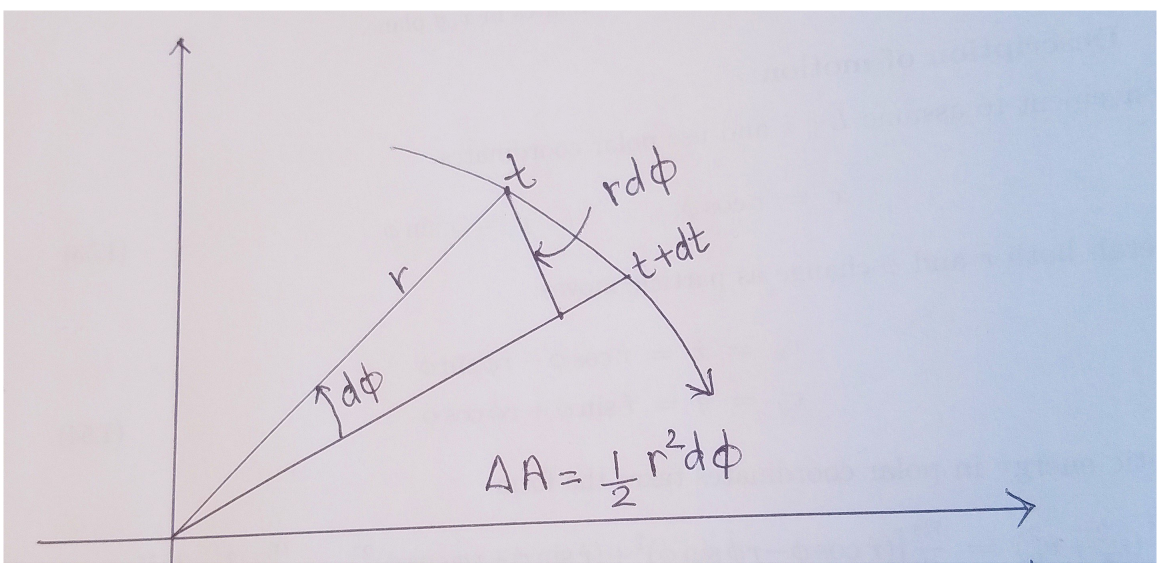

Figure 2: Area swept out by the radius vector **r** in time dt

$$dA = \frac{1}{2}r\ (r\ d\varphi) = \frac{1}{2}r^2\dot{\varphi}\ dt$$
$$\therefore \dot{A} = \frac{dA}{dt} = \frac{1}{2}r^2\dot{\varphi} = \frac{l}{2\mu} = \text{constant} \tag{14}$$

Thus, the particle sweeps away equal area in equal interval of time, which is **Kepler's second law**. Again

$$\int_{\varphi_1}^{\varphi_2} d\varphi = \varphi_2 - \varphi_1 = \frac{l}{\mu r^2}\int_{t_1}^{t_2} dt = \frac{l}{\mu r^2}(t_2 - t_1) \tag{15}$$

from Eq. (12) and Eq. (14), we get

$$\varphi_2 - \varphi_1 = \frac{l}{\mu}\int_{r_1}^{r_2} \frac{dr}{\sqrt{\frac{2}{\mu}}r^2\sqrt{E - \frac{P_\varphi^2}{2\mu r^2} - V(r)}}$$
$$\varphi_2 - \varphi_1 = \frac{l}{\sqrt{2\mu}}\int_{r_1}^{r_2} \frac{dr/r^2}{\sqrt{E - \frac{P_\varphi^2}{2\mu r^2} - V(r)}} \tag{16}$$

Again using Eq. (7), we can write

$$d\varphi = \frac{l}{\mu r^2}dt$$
$$\frac{d}{dt} = \frac{l}{\mu r^2}\frac{d}{d\varphi}$$
$$\dot{r} = \frac{dr}{dt} = \frac{l}{\mu r^2}\frac{dr}{d\varphi}$$
$$\ddot{r} = \frac{l}{\mu r^2}\frac{d\dot{r}}{d\varphi} = \frac{l}{\mu r^2}\frac{d\dot{r}}{d\varphi} = \frac{l}{\mu r^2}\frac{d}{d\varphi}\left(\frac{l}{\mu r^2}\frac{dr}{d\varphi}\right) \tag{17}$$

Again Let $r = \frac{1}{u}$

$$\frac{dr}{d\varphi} = \frac{dr}{du}\frac{du}{d\varphi}$$
$$= \frac{-1}{u^2}\frac{du}{d\varphi}$$
$$= -r^2\frac{du}{d\varphi}$$
$$\frac{dr}{r^2} = -du \tag{18}$$

Using Eq.(15) and Eq.(17), we get, taking + sign ,

$$\varphi_2 - \varphi_1 = \frac{l}{\sqrt{2\mu}}\int \frac{du}{\sqrt{E - \frac{l^2u^2}{2\mu} - V(u)}} \tag{19}$$

Using Eq.(16) in Eq.(9),

$$\frac{-l^2u^2}{\mu}\left(\frac{d^2u}{d\phi^2} - \frac{l^2u^3}{\mu} + u^2\frac{dV}{du}\right) = 0$$

$$\frac{d^2u}{d\phi^2} = -u + \frac{\mu}{l^2}\frac{dV}{\partial u} \tag{20}$$

Considering the power law function of r for the potential such that

$$V(r) = k\ r^{n+1} \tag{21}$$
$$V(u) = k\ u^{-(n+1)} \tag{22}$$

Eq. (18) becomes

$$d\varphi = \int \frac{du}{\sqrt{\frac{2\mu E}{l^2} - \frac{2\mu k}{l^2} u^{-(n+1)} - u^2}} \tag{23}$$

Again,

$$E = \frac{\mu}{2}\dot{r}^2 + \frac{l^2}{2\mu r^2} + V(r) = \frac{\mu}{2}\dot{r}^2 + \text{"}V(r)\text{"}$$

At $r = r_{min}$, $r = r_{max}$, and $r = r_0$, equilibrium position $\dot{r} = 0$
For equilibrium position $r = r_0$,

$$\frac{\partial \text{"}V(r)\text{"}}{\partial r} = 0 \qquad \text{and} \quad E = E_0 \tag{24}$$

For a mass μ on a spring with spring constant k_s, $V = \frac{k_s}{2} r^2$, so $k = k_s/2$ and $n = 1$. For Kepler's problem , $n = -2$, $k = GM\mu$, and $V(u) = -ku$:

$$\frac{-l^2}{2\mu r_0^3} + \frac{k}{r_0^2} = 0$$

$$r_0 = \frac{l^2}{\mu k} \tag{25}$$

and

$$E_0 = \frac{l^2}{2\mu}\left(\frac{\mu k}{l^2}\right)^2 - k\frac{\mu k}{l^2}$$
$$E_0 = \frac{-\mu k^2}{2l^2} = V/2 = -T \tag{26}$$

[Note: Alternative way to find maximum and minimum values of r (From Goldstein Text)
For maximum and minimum values of r ,

$$E = \frac{l^2}{2\mu r^2} - \frac{k}{r}$$

$$Er^2 + kr - \frac{l^2}{2\mu} = 0$$

This equation is quadratic in r, so we will have two roots given by:

$$\begin{aligned} r &= \frac{-k \pm \sqrt{k^2 + \frac{2El^2}{\mu}}}{2E} \\ &= \frac{-k}{2E}\left(1 \pm \sqrt{1 + \frac{2El^2}{\mu k^2}}\right) \\ &= a(1 \pm e) \end{aligned}$$

with,

$$a = \frac{-k}{2E} \quad \text{and} \quad e = \sqrt{1 + \frac{2El^2}{\mu k^2}} \qquad]$$

We get From Eq. (19)

$$u'' + u = \frac{\mu k}{l^2}$$
$$u(\varphi) = \frac{1}{r} = Acos(\varphi - \varphi_0) + \frac{\mu k}{l^2}$$

$$\begin{aligned} r &= \frac{1}{Acos(\varphi - \varphi_0) + \frac{\mu k}{l^2}} \\ &= \frac{1}{Acos(\varphi - \varphi_0) + C} \\ \text{therefore,} \quad r &= \frac{1}{C(1 + e\ cos(\varphi - \varphi_0))} \end{aligned} \tag{27}$$

Without loss of generality, let us assume that $\varphi_0 = 0$ at $t = 0$, so the above Eq. (30) becomes

$$r = \frac{1}{C(1 + e\ cos\varphi)} \tag{28}$$

where $C = \frac{\mu k}{l^2}$ and $e = \frac{A}{C}$ is the eccentricity of the orbit of the particle.

Now, for $r = r_{min}$

$$\begin{aligned} E &= \frac{l^2}{2\mu r_{min}^2} - \frac{k}{r_{min}} \\ &= \frac{l^2}{2\mu} C^2 (1+e)^2 - kC(1+e) \\ E &= \frac{\mu k^2}{2l^2}(1 - e^2) \\ e &= \sqrt{1 + \frac{2El^2}{\mu k^2}} \end{aligned} \tag{29}$$

At equilibrium

$$E_0 = \frac{-\mu k^2}{2l^2}$$

The nature of the orbit depends upon the magnitude of e according to the following scheme:

$e = 0,$	$E = -\frac{\mu k^2}{2l^2}:$	circle
$e = 1,$	$E = 0:$	parabola
$e > 1,$	$E > 0:$	hyperbola
$e < 1,$	$E < 0:$	ellipse

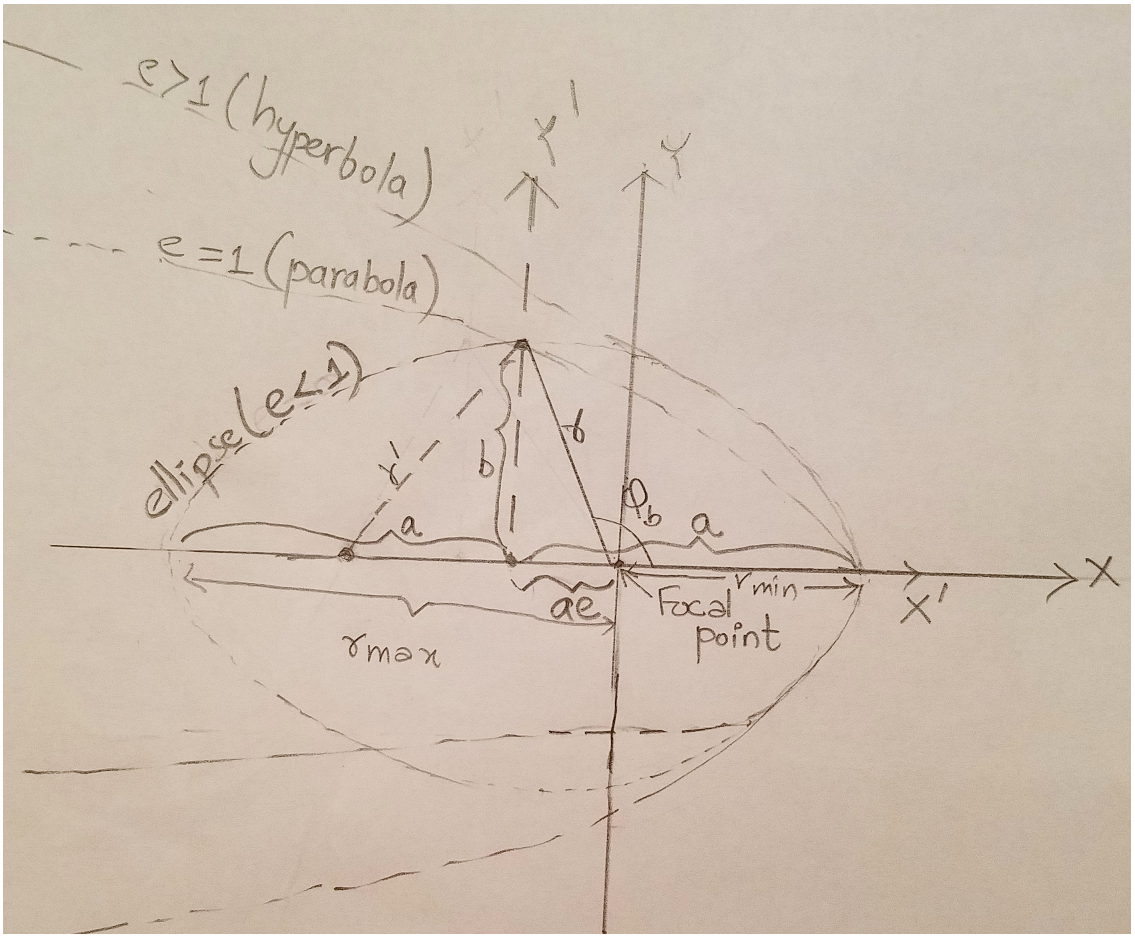

Figure 3: Trajectory of the body with varying eccentricity in the central force field

For $e < 1$, case of ellipse,

$$r_{min} = \frac{1}{C(1+e)}$$
$$r_{max} = \frac{1}{C(1-e)}$$

The major half axis, a is defined by the relation

$$\begin{aligned} 2a &= r + r' \\ 2a &= r_{min} + r_{max} \\ a &= \frac{1}{2C}\left(\frac{1}{1+e} + \frac{1}{1-e}\right) \\ a &= \frac{1}{C(1-e^2)} \\ &= -\frac{k}{2E} = \frac{1}{1-e^2}\frac{l^2}{\mu k} \end{aligned} \tag{30}$$

So, Choose φ_0 such that $r(\varphi_0) = r_{min}$

$$r = \frac{1}{C(1 + e\ cos\varphi)} = \frac{a(1 - e^2)}{(1 + e\ cos\varphi)} \tag{31}$$

From Fig.(3),

$$r(\varphi_b)(-cos\varphi_b) = a\ e$$

$$\text{or,}\quad -\frac{a(1 - e^2)cos\varphi_b}{1 + ecos\varphi_b} = ae$$

$$\text{or,}\qquad -(e + e^2cos\varphi_b) = (1 - e^2)cos\varphi_b$$

$$\text{or,}\quad cos\varphi_b = -e$$

$$\therefore\qquad r_b(\varphi_b)(-cos\varphi_b) = r_b e = ae \Rightarrow r_b = a \tag{32}$$

Then, we get,

$$b = \sqrt{r_b^2 - a^2e^2} = a\sqrt{1 - e^2} = \frac{1}{\sqrt{1 - e^2}}\frac{l^2}{\mu k} = \sqrt{a}\sqrt{\frac{l^2}{\mu k}} \tag{33}$$

The equation

$$r = \frac{1}{C(1 + e\ cos\varphi)} \tag{34}$$

is actually an equation of an ellipse with shifted co-ordinates x' and y' (or x and y, original co-ordinate system)

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1 \tag{35}$$

$$\frac{(x + x_0)^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{36}$$

with,

$$x_o = ae, \qquad a = \frac{1}{C(1 - e^2)}, \qquad b = \frac{1}{C\sqrt{1 - e^2}} \tag{37}$$

This can be proven by using $y' = y = r(\varphi)\sin(\varphi)$ and $x' = ae + r(\varphi)\cos(\varphi)$ and plugging in.

Kepler's Third Law

Now area of ellipse $A = \pi ab$

The period of elliptical motion T is the ratio of the total area of the ellipse (A) to the areal velocity ($\dot{A}$) and is given as :

$$T = \frac{\pi ab}{l/2\mu} = \frac{2\pi\mu ab}{l}$$

$$T = 2\pi a^{3/2}\sqrt{\frac{\mu}{k}} = 2\pi a^{3/2}\sqrt{\frac{1}{GM}}$$

$$T^2 = 4\pi^2 a^3\frac{\mu}{k} \tag{38}$$

Because

$$b^2 = a^2\sqrt{1 - e^2} = \left(\frac{-k}{2E}\right)^2 . \frac{-2El^2}{\mu k^2} = \frac{-k}{2E}.\frac{l^2}{\mu k}$$

$$b = a^{1/2}\sqrt{\frac{l^2}{\mu k}} \tag{39}$$

The Eq.(38) shows that the square of the periods of the object in central force is proportional to the cube of the major half axis i. e $T^2 \propto a^3$, which is **Kepler's third law**.

[Note: If a planetory object of mass m is in the motion under the potential of central force, we should replace the reduced mass μ by mass m of the planet]