Recall the energy equation from the last lecture:

$$h = \frac{\mu}{2} \dot{r}^2 + \frac{P_\phi^2}{2\mu r^2} + V(r) = E \quad (1)$$

Also recall that we designated

"$$V(r)" = \frac{P_\phi^2}{2\mu r^2} + V(r)$$

Now we’ll bring in the concept of gravitational attraction:

$$V(r) = -\frac{G M \mu}{r} = -\frac{k}{r}$$

For brevity’s sake let $$k = GM\mu$$.

We’ll assume all motion is in the x-y plane. Rearranging $$\dot{r}$$:

$$\dot{r} = \frac{dr}{d\phi} \dot{\phi} = r' \frac{P_\phi}{\mu r^2} = \frac{P_\phi}{\mu} \left(-\frac{\partial}{\partial \phi} \frac{1}{r}\right)$$

Now make the variable transformation

$$u = \frac{1}{r} \Rightarrow \dot{r} = \frac{P_\phi}{\mu} u'$$

Now rewriting (1):

$$\frac{P_\phi^2}{2\mu} u'^2 + \frac{P_\phi^2}{2\mu} u^2 - ku = E$$

Rearranging:

$$u'^2 = \frac{2\mu}{P_\phi^2} (E + ku) - u^2$$
Now substituting this expression into the integral from the previous lecture:

\[
\int_{u(\phi_0)}^{u(\phi)} \frac{du}{\sqrt{\frac{2\mu E}{P_\phi^2} + \frac{2\mu k}{P_\phi^2} u - u^2}} = \int_{\phi_0}^{\phi} d\phi'
\]

Note that \( E \to \infty \) as \( r \to 0 \). Also note we are assuming \( P_\phi \neq 0 \). The maximum value of \( u \) occurs where \( u' = 0 \). Designate this value \( u_m \). With \( u' = 0 \) our energy expression becomes:

\[
\frac{2\mu}{P_\phi^2} (E + ku_m) - u^2 = 0
\]

Completing the polynomial square:

\[
\frac{2\mu E}{P_\phi^2} + \frac{\mu^2 k^2}{P_\phi^4} - (u_m - \frac{\mu k}{P_\phi^2})^2 = 0
\]

Solving for \( u_m \):

\[
u_m = \frac{\mu k}{P_\phi^2} \pm \sqrt{\frac{2\mu E}{P_\phi^2} + \frac{\mu^2 k^2}{P_\phi^4}}
\]

Where the max occurs with the + sign active in the expression. We’ll set \( u(\phi_0) = u_m \) and \( \phi_0 = 0 \), so \( u(0) = u_m \).

Now returning to the integral. Define \( v = u - \frac{\mu k}{P_\phi^2} \). Rewriting the integral:

\[
\int_{u_m - \frac{\mu k}{P_\phi^2}}^{u(\phi)} \frac{dv}{\sqrt{\frac{2\mu E}{P_\phi^2} + \frac{\mu^2 k^2}{P_\phi^4} - v^2}} = \int_{\phi_0}^{\phi} d\phi'
\]

Now define \( w = \frac{v}{\sqrt{\frac{2\mu E}{P_\phi^2} + \frac{\mu^2 k^2}{P_\phi^4}}} \). And again rewriting the integral:

\[
\int_{1}^{w(\phi)} \frac{dw}{\sqrt{1 - w^2}} = \int_{\phi_0}^{\phi} d\phi'
\]

Reversing the limits of integration:

\[
-\int_{w(\phi)}^{1} \frac{dw}{\sqrt{1 - w^2}} = \int_{\phi_0}^{\phi} d\phi'
\]

\(
\arccos(1) - \arccos(w(\phi)) = \phi
\)
\[
\cos \phi = \frac{u(\phi) - \frac{\mu k}{P^2_\phi}}{\sqrt{\frac{2\mu E}{P^2_\phi} + \frac{\mu^2 k^2}{P^2_\phi}}} \\
\]
\[
u(\phi) = \frac{\mu k}{P^2_\phi} (1 + \cos \sqrt{1 + \frac{2EP^2_\phi}{\mu k^2}}) \\
\]
Now define \( c = \frac{\mu k}{P^2_\phi} \) and \( e = \sqrt{1 + \frac{2EP^2_\phi}{\mu k^2}} \), making the expression now:
\[
u(\phi) = c(1 + e \cos \phi) \\
\]
Recall \( u = \frac{1}{r} \):
\[
r(\phi) = \frac{1}{c(1 + e \cos \phi)} \\
\]
Now we’ll explore the impact manipulating the variable \( e \) has on the shape of the function \( r(\phi) \). Note they are all conic sections.

**Case 1**
\( e = 0 \)
Circle
\[
r(\phi) = \frac{1}{c} \\
c = \frac{\mu k}{P^2_\phi} \\
\]
recall \( k = GM\mu \):
\[
r(\phi) = \frac{1}{c} = \frac{P^2_\phi}{\mu k} = \frac{P^2_\phi}{GM\mu^2} \\
\]
Note \( r(\phi) \) is a constant, and since \( \dot{\phi} = \frac{P_\phi}{\mu r^2} \Rightarrow \dot{\phi} \) is constant.
And since \( e = \sqrt{1 + \frac{2EP^2_\phi}{\mu k^2}} \), \( e = 0 \Rightarrow E = -\frac{\mu k}{2P^2_\phi} \), the minimum energy the system can have without a complex solution.

**Case 2**
\( e = 1 \)
Parabola
\[
r(\phi) = \frac{1}{c(1 + e \cos \phi)} = \frac{1}{c(1 + \cos \phi)} \\
e = \sqrt{1 + \frac{2EP^2_\phi}{\mu k^2}}; \quad e = 1 \Rightarrow E = 0 \\
\]
Case 3

$e > 1$

Hyperbola

\[ r(\phi) = \frac{1}{c(1 + e \cos \phi)} \]

\[ e = \sqrt{1 + \frac{2EP^2}{\mu k^2}}; \quad e > 1 \Rightarrow E > 0 \]

More to follow on the hyperbola in the next lecture.

Case 4

$0 < e < 1$

Ellipse

\[ r(\phi) = \frac{1}{c(1 + e \cos \phi)} \]

\[ r_{\min} = \frac{1}{c(1 + e \cos \phi)}; \quad r_{\max} = \frac{1}{c(1 - e \cos \phi)} \]

Solving for the semi-major axis, $a$:

\[ 2a = \frac{1}{c(1 - e)} + \frac{1}{c(1 + e)} \]

\[ a = \frac{1}{c(1 - e^2)} \]

Semi-minor axis, $b$:

\[ b = a\sqrt{1 - e^2} \]