Classical Mechanics - Problem Set 1 - Solution

Problem 1)

There is only one degree of freedom left: the number of turns or, more precisely, the total angle (including an increase of $2\pi$ for each turn!) by which the marble had to be rotated around the axis to move from the outer rim to its present position. Therefore, we define this angle $\Phi$ as our generalized coordinate.

To express $x$ and $y$ in this coordinate, it is useful to first express the radial distance $r$ from the record axis in terms of $\Phi$: $r(\Phi) = R_2 - \frac{R_2 - R_1}{2\pi N} \Phi$.

Now we can simply convert polar into cartesian coordinates:

$$x = r(\Phi) \cos \Phi = \left[ R_2 - \frac{R_2 - R_1}{2\pi N} \Phi \right] \cos \Phi$$

$$y = r(\Phi) \sin \Phi = \left[ R_2 - \frac{R_2 - R_1}{2\pi N} \Phi \right] \sin \Phi$$

From this, we can derive the components of the cartesian velocity:

$$\dot{x} = \left( \frac{\partial r(\Phi)}{\partial \Phi} \cos \Phi + r(\Phi) \frac{\partial \cos \Phi}{\partial \Phi} \right) \dot{\Phi} = \left( -\frac{R_2 - R_1}{2\pi N} \cos \Phi - \left[ R_2 - \frac{R_2 - R_1}{2\pi N} \Phi \right] \sin \Phi \right) \dot{\Phi}$$

$$\dot{y} = \left( \frac{\partial r(\Phi)}{\partial \Phi} \sin \Phi + r(\Phi) \frac{\partial \sin \Phi}{\partial \Phi} \right) \dot{\Phi} = \left( -\frac{R_2 - R_1}{2\pi N} \sin \Phi + \left[ R_2 - \frac{R_2 - R_1}{2\pi N} \Phi \right] \cos \Phi \right) \dot{\Phi}$$

The kinetic energy of the marble then is

$$T = \frac{m}{2} (\dot{x}^2 + \dot{y}^2) = \frac{m}{2} \left( \left[ \frac{R_2 - R_1}{2\pi N} \right]^2 + \left[ \frac{R_2 - R_2 - R_1}{2\pi N} \Phi \right]^2 \right) \dot{\Phi}^2$$

Finally, we can write for the generalized force

$$Q_\Phi = \left( F_x \frac{\partial x}{\partial \Phi} + F_y \frac{\partial y}{\partial \Phi} \right)$$

$$= \left( \left[ -\frac{R_2 - R_1}{2\pi N} \cos \Phi - \left[ R_2 - \frac{R_2 - R_1}{2\pi N} \Phi \right] \sin \Phi \right] F_x + \left( -\frac{R_2 - R_1}{2\pi N} \sin \Phi + \left[ R_2 - \frac{R_2 - R_1}{2\pi N} \Phi \right] \cos \Phi \right) F_y \right)$$
Finally, we get the following two equations of motion:

\[
\frac{d}{dt} \frac{\partial T}{\partial \dot{\Phi}} - \frac{\partial T}{\partial \Phi} = m \frac{d}{dt} \left( \left[ \frac{R_2 - R_1}{2\pi N} \right]^2 \dot{\Phi} + \left[ R_2 - \frac{R_2 - R_1}{2\pi N} \Phi \right]^2 - m \left[ R_2 - \frac{R_2 - R_1}{2\pi N} \Phi \right] \left( - \frac{R_2 - R_1}{2\pi N} \right) \dot{\Phi} \right)
\]

\[
m \left( \left[ \frac{R_2 - R_1}{2\pi N} \right]^2 + \left[ R_2 - \frac{R_2 - R_1}{2\pi N} \Phi \right]^2 \right) \ddot{\Phi} + 2m \left[ R_2 - \frac{R_2 - R_1}{2\pi N} \Phi \right] \ddot{\Phi} - m \left[ R_2 - \frac{R_2 - R_1}{2\pi N} \Phi \right] \dddot{\Phi} = Q_\Phi
\]

**Problem 2)**

The equations of motion are then given by

\[
m \ddot{x} = F_x ; \quad m \ddot{y} = F_y
\]

The coordinates \(q_1\) and \(q_2\) are simply the original cartesian coordinates, rotated by a time-dependent angle \(\omega t\). We can express \(x, y\) in these coordinates by applying the inverse rotation:

\[
x = q_1 \cos \omega t - q_2 \sin \omega t
\]

\[
y = q_1 \sin \omega t + q_2 \cos \omega t
\]

(check that this is compatible with the definitions of the \(q_i\)'s!). The time derivatives in terms of the new coordinates are

\[
\dot{x} = \dot{q}_1 \cos \omega t - \dot{q}_2 \sin \omega t - \omega q_1 \sin \omega t - \omega q_2 \cos \omega t
\]

\[
\dot{y} = \dot{q}_1 \sin \omega t + \dot{q}_2 \cos \omega t + \omega q_1 \cos \omega t - \omega q_2 \sin \omega t
\]

Squaring and adding drops several cross terms and one gets

\[
\dot{x}^2 + \dot{y}^2 = \dot{q}_1^2 + \omega^2 q_1^2 + \dot{q}_2^2 + \omega^2 q_2^2 - 2\dot{q}_1 \omega q_2 + 2\omega q_1 \dot{q}_2 = (\dot{q}_1 - \omega q_2)^2 + (\dot{q}_2 + \omega q_1)^2
\]

which yields a new expression for the kinetic energy:

\[
T = \frac{m}{2} \left[ (\dot{q}_1 - \omega q_2)^2 + (\dot{q}_2 + \omega q_1)^2 \right]
\]

The generalized forces are simply the rotated cartesian forces:

\[
Q_1 = F_x \frac{\partial x}{\partial q_1} + F_y \frac{\partial y}{\partial q_1} = F_x \cos \omega t + F_y \sin \omega t ; \quad Q_2 = -F_x \sin \omega t + F_y \cos \omega t
\]

Finally, we get the following two equations of motion:
\[\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = \frac{d}{dt} m(\dot{q}_1 - \omega q_2) - m(\dot{q}_2 + \omega q_1) \omega = m\ddot{q}_1 - 2m\omega \dot{q}_2 - m\omega^2 q_1 = Q_1 \]
\[\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_2} - \frac{\partial T}{\partial q_2} = \frac{d}{dt} m(\dot{q}_2 + \omega q_1) + m(\dot{q}_1 - \omega q_2) \omega = m\ddot{q}_2 + 2m\omega \dot{q}_1 - m\omega^2 q_2 = Q_2\]

In addition to the force components in the direction of the new variables, \(Q_1\) and \(Q_2\), we also have the “Coriolis force” (the middle terms on the left-hand sides) and the “centrifugal forces” \((m\omega^2 q)\) contributing to the acceleration in the direction of \(q_1\) and \(q_2\).

**Problem 3**

According to Eq. 1.31, we can express the kinetic energy as follows:
\[T = \frac{M}{2} \ddot{V}^2 + \frac{m_1}{2} \dot{v}_1^2 + \frac{m_2}{2} \dot{v}_2^2.\]
We can write
\[
\ddot{v}_1' = \ddot{v}_1 - \ddot{v}_2 = \frac{m_1 \ddot{v}_1 + m_2 \ddot{v}_2}{M} = \frac{m_2 \ddot{v}_1 - m_1 \ddot{v}_2}{M} = \frac{-m_2 \ddot{v}}{M} \quad \text{and similarly} \quad \ddot{v}_2' = \ddot{v}_2 - \ddot{v}_1 = \frac{m_1 \ddot{v}}{M}.
\]
This yields for the kinetic energy
\[T = \frac{M}{2} \ddot{V}^2 + \frac{m_1}{2} \frac{m_2^2}{M^2} \dot{v}_1^2 + \frac{m_2}{2} \frac{m_1^2}{M^2} \ddot{v}_2^2 = \frac{M}{2} \ddot{V}^2 + \frac{m_1 m_2 (m_2 + m_1)}{2M^2} \ddot{v}_1^2 = \frac{M}{2} \ddot{V}^2 + \frac{m_1 m_2}{2M} \ddot{v}_2^2.\]
If we identify the expression \(\frac{m_1 m_2}{M}\) with the new “reduced mass” \(\mu\), we get the form given in the problem.

**Problem 4**

After the collision, the ball and the left mass \(m_1\) join into a new mass \(M_1 = m_1 + m\) with initial position \(x_1 = 0\) and initial velocity \(v_1 = \nu m/M_1\) (momentum conservation). Hence
\[X = (x_1 M_1 + x_2 m_2)/M\]
is the center-of-mass coordinate \((M = M_1 + m_2)\) with velocity \(V = \nu m/M = \text{const.}\) (since there are no external forces acting on the system after the collision) and the relative coordinate is \(x = x_2 - x_1\) (initially \(x(t=0) = L\)). The relative velocity \(\nu = dx/dt\) is initially equal to \(-v_1 = -\nu m/M_1\).

Ignoring the c.m. kinetic energy (the first term in the last expression for \(T\) from Problem 3), the equivalent one-body problem is that with a (reduced) mass of \(\mu = M_1 m_2 / M\) and kinetic energy \(T = \mu/2 \nu^2\). The potential energy is due to the deformation of the spring and is equal to \(V = k/2 (x-L)^2\). This is just an ordinary harmonic oscillator with the solution
\[ x = L + A \cos(\omega t) + B \sin(\omega t) \] where \( \omega = \sqrt{k/\mu} \). Since \( x = L \) at \( t = 0 \), the constant \( A \) must be zero. Taking the derivative, we find that \( v = \omega B \cos(\omega t) \) which must be equal to \(-\nu m/M_1 \) at \( t = 0 \). Hence, \( B = -\nu m/\omega M_1 \). Therefore, the coordinate \( x \) makes oscillations with amplitude \( B \) around the equilibrium position \( L \) with angular frequency \( \omega \).

**Problem 5)**

1. The momentum of the combined system “Rocket” and “Spent Fuel” will change with time according to the equation
\[
\frac{dP_x}{dt} = \frac{dp_R}{dt} + \frac{dp_E}{dt} = \left( M_R + M_F(t) \right) \frac{d\dot{v}}{dt} + \left( -\frac{dM_F(t)}{dt} \right) (-u) = \left( M_R + M_F(t) \right) \dot{v} - u\gamma = -A\nu^2
\]
which leads to \( \dot{v} = \frac{u\gamma - A\nu^2}{M_R + M_F(t)} \).

2. Obviously, it must be equal to \( A\nu^2 \).

3. In that case, \( \gamma = 0 \) and \( \dot{v} = -\frac{A\nu^2}{M_R} \Rightarrow \frac{1}{\nu(t)} - \frac{1}{\nu_0} = \frac{A}{M_R} t \Rightarrow v(t) = \left( \frac{A}{M_R} t + \frac{1}{\nu_0} \right) \nu_0 \)

4. During a short time interval \( \Delta t \), the number of particles hitting the nose cone is \( \Delta n = \pi R^2 N \nu \Delta t \). In the rest frame of the rocket, all particles get deflected at 90 degrees, meaning their initial velocity changes from \(-\nu \) to zero in x-direction. (We can ignore the sideways motions of the particles after the collision since all of them together will average out to zero). Since the CHANGE of momentum is independent of the coordinate system, we can conclude that the momentum of these particles combined changes by \( \Delta p = \pi R^2 N \nu \Delta t \). Momentum conservation tells us that the resulting force on the rocket is \( F = -\Delta p / \Delta t = -\pi R^2 N \nu \Delta t \) which has the claimed form and \( A = \pi R^2 N \).