Classical Mechanics - Problem Set 1 – Due January 23

Problem 1)

Consider a vinyl record (now once more fashionable!). Assume it has a single groove that spirals inward from the edge at radius $R_2$ to the inner radius $R_1$. There are $N$ turns, all evenly spaced between $R_1$ and $R_2$. Assume I put a (tiny) marble of mass $m$ into the groove, moving inside the groove frictionless without rolling. Obviously, it has only one degree of freedom (it can move along the groove, but not perpendicular to it). Use as generalized coordinate the angle $\Phi$ between the starting point (at the outer edge, radius $R_2$) and the instantaneous position of the marble (increasing by $2\pi$ for every full turn). Express the cartesian coordinates $x$ and $y$ describing the marble position relative to a fixed coordinate system centered on the record axis in terms of that generalized coordinate. (For the time being, assume the record does not rotate).

Using the same generalized coordinate, write down the (cartesian) velocity of the marble, and the kinetic energy. Find the generalized force $Q_{\Phi}$ in terms of the force components $F_x$ and $F_y$ in $x$- and $y$-direction. What are the equations of motion for the marble? (Don’t attempt to solve them).

Problem 2)

The position of a particle of mass $m$ moving freely in two dimensions in an inertial Cartesian coordinate system is given by $(x,y)$. Write down the particle’s kinetic energy $T(\dot{x}, \dot{y})$ and the equations of motion. Generalized coordinates $q_1(x, y, t)$ and $q_2(x, y, t)$ are defined by

$$q_1 = x \cos \omega t + y \sin \omega t$$

$$q_2 = -x \sin \omega t + y \cos \omega t$$

where $\omega$ is a constant. Write down the inverse transformations, $x(q_1,q_2,t)$ and $y(q_1,q_2,t)$ and hence find an expression for $T(q_1,q_2,\dot{q}_1,\dot{q}_2, t)$ in the new coordinates. Obtain the equations of motion in the new coordinates (including the generalized forces $Q_1$ and $Q_2$). Identify the extra terms that appear, in terms of so-called “fictitious forces”.

Problem 3)

On page 10 in our book, it is shown how one can split up the total kinetic energy of a system of mass points into a part involving the motion of the center of mass only ($1/2 \ M \ V^2$; $V = dR/dt$) and a term involving the “internal” velocities $\mathbf{v}'_i = \mathbf{v}_i - \mathbf{V}$ (see Eq. 1.31). Of course, the $\mathbf{v}'_i$ are not all independent, since $\sum_i m_i \dot{v}'_i = 0$ by the definition of the center of mass. In particular, for a system of only two particles, we can replace $\mathbf{v}_1'$ and $\mathbf{v}_2'$ by a single variable, $\mathbf{v} = d\mathbf{r}/dt$. Here, $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = \mathbf{r}_2' - \mathbf{r}_1'$ is the relative coordinate (the distance
vector) between the two mass points. Show that in this case, you can write the kinetic energy with just two terms involving only $V$ and $v$: $T = \frac{M}{2} \dot{V}^2 + \frac{\mu}{2} \dot{v}^2$. Express the “reduced mass” $\mu$ in terms of the two masses $m_1$ and $m_2$.

**Problem 4) (Application of Problem 3)**

Two balls of mass $m_1$ and $m_2$ are connected by a spring with an elastic constant $k$. Initially, $m_1$ is located at $x = 0$ and $m_2$ at $x = L$ which corresponds to the relaxed length of the spring. A third ball of mass $m$ moving with a velocity $v$ from the left hits ball 1 and gets instantly stuck to it, as shown in the figure below. Assuming that the balls 1 and 2 were initially at rest and can slide without friction only along the $x$-axis, calculate the amplitude and frequency of oscillations after the impact. Use the results and nomenclature of Problem 3 (2\textsuperscript{nd} half) to solve this problem.

![Diagram of two balls connected by a spring](image)

**Problem 5)**

A cylindrical rocket of diameter $2R$, mass $M_R$ and containing fuel of mass $M_F$ is moving through empty space at velocity $v_0$. At some point the rocket enters a uniform cloud of interstellar particles with volume density $N$ (e.g., particles/m$^3$), each particle having the mass $m \ll M_R$ and initially at rest. To compensate for the dissipative force of the particles colliding with the rocket, the rocket engines emit fuel at the rate $\gamma$ with $dM_F/dt = -\gamma$ at a constant velocity $u$ with respect to the rocket. Ignore gravitational interaction between the rocket and the cloud particles.

1. Assuming that the dissipative drag force from the cloud particles is $-Av^2$, where $A$ is a constant, derive the differential equation for the velocity $v(t)$ of the rocket through the cloud as it is firing its engines.

2. What must the rocket thrust $\gamma u$ be to maintain the constant velocity $v_0$?
3. If the rocket suddenly runs out of fuel, how would its velocity \( v(t) \) decrease with time as the rocket moves through the cloud?

4. BONUS: Assuming that each cloud particle bounces off the rocket elastically, and collisions occur very frequently, prove that the drag force is indeed proportional to \( v^2 \) and estimate the constant \( A \), if the front nose-cone of the rocket has the opening angle of 90°.