Problem 1)

(1) Since the energy of the $Z^0$ at rest is $mc^2 = 90$ GeV, each electron must have an energy of exactly 45 GeV (the spatial components of the momenta add up to zero in a head-on collision). Given the mass of an electron or positron is $0.000511$ GeV/c$^2$, the gamma factor must be $88,063$. This corresponds to a velocity of $u = 0.9999999999355$ c (note that you must have a pretty good calculator to get this right – there are 10 9’s before the first digit that is not a 9!). The momentum can be calculated as either $\vec{p} = \gamma mc^2 / c$ or directly as $\vec{p} = \gamma mc = \frac{E}{c}$.

(2) This is most easily solved using the invariant (rest frame) energy of the final state, which is equal to

$$E_{\text{inv}} = 90 \text{ GeV} = \sqrt{E_{\text{tot}}^2 - (p_{\text{tot}} c)^2} = \sqrt{\left(E_e + m_e c^2\right)^2 - \left(\frac{E_e}{c}\right)^2 - \left(m_e c^2\right)^2}$$

where I used the fact that the total spatial part of the momentum is equal to the momentum of the positron alone, given that the electron is at rest. Squaring both sides and solving for $E_e$ yields

$$E_e = \frac{E_{\text{inv}}^2}{2m_e c^2} - m_e c^2 = 7.92564 \cdot 10^6 \text{ GeV} = 7.9 \text{ PeV} \times 10^{15} \text{ eV}$$

which is about 1000 times more energy than the largest accelerator in the world, the LHC, can provide (and not to electrons!), or nearly a million times more than CEBAF at Jefferson Lab. Clearly, colliders are the way to go!

Problem 2)

The initial total 4-momentum vector is $p_{\text{tot}}^\mu = p_T^\mu + (m_e c, 0, 0, 0) = \left(\frac{E_1}{c}, m_e c, 0, \frac{E_1}{c}\right)$ where

I have chosen the initial photon direction along the z-axis. Let the final photon momentum lie in the x-z plane and be given by $p_T^\mu = \left(\frac{E_2}{c}, \frac{E_2}{c} \sin \theta, 0, \frac{E_2}{c} \cos \theta\right)$. Then the final
electron momentum must be \( p_{e,f}^\mu = p_{\text{tot}}^\mu - p_{r,f}^\mu = \left( \frac{E_1}{c} + m_e c - \frac{E_2}{c}, -\frac{E_2}{c} \sin \theta, 0, \frac{E_1}{c} - \frac{E_2}{c} \cos \theta \right) \).

We can impose the additional requirement that the invariant square of this final electron momentum must equal to the mass (times \( c \)) squared:

\[
p_{e,f}^\mu p_{e,f}^\mu = \left( \frac{E_1}{c} + m_e c - \frac{E_2}{c} \right)^2 - \left( -\frac{E_2}{c} \sin \theta \right)^2 - \left( \frac{E_1}{c} - \frac{E_2}{c} \cos \theta \right)^2 = \left( \frac{E_1}{c} + m_e c \right)^2 - 2 \left( \frac{E_1}{c} + m_e c \right) \frac{E_2}{c} - \left( \frac{E_1}{c} \right)^2 + 2 \frac{E_1}{c} \frac{E_2}{c} \cos \theta = 2 \frac{E_1}{c} m_e c + m_e^2 c^2 - \frac{E_2}{c} \left( 2 \frac{E_1}{c} \left( 1 - \cos \theta \right) + 2 m_e c \right)^2 \Rightarrow \]

\[
E_z = \frac{2 E_1 m_e c}{2 \frac{E_1}{c} (1 - \cos \theta) + 2 m_e c} = \frac{m_e^2 c^2}{E_1 (1 - \cos \theta) + m_e^2 c^2} E_i
\]

Given that the wavelength of a photon is \( \lambda = \frac{hc}{E} \), we can also write this as

\[
\lambda_\gamma = \frac{hc}{E_2} \frac{E_1}{E_i} \frac{1 - \cos \theta + 1}{m_e c^2 (1 - \cos \theta) + 1} = \frac{h}{m_e c} (1 - \cos \theta) + \lambda_1 \) (the expression \( h/m_e c \) is called the “Compton wavelength of the electron”).

**Problem 3)**

1) The Hamilton equations of motion are as follows:

\[
\dot{r} = \frac{\partial H}{\partial P_r} = \frac{P_r c^2}{E}, \quad \dot{\phi} = \frac{\partial H}{\partial P_\phi} = \frac{\left( \frac{P_\phi}{r} - q \frac{r B}{2} \right) c^2}{r}, \quad \dot{z} = \frac{\partial H}{\partial P_z} = \frac{P_z c^2}{E} = \frac{P_z}{\gamma m} \]

\[
\dot{r} = -\frac{\partial H}{\partial r} = -\frac{\left( \frac{P_\phi}{r^2} - q \frac{r B}{2} \right) \left( \frac{P_\phi}{r^2} - q \frac{B}{2} \right) c^2}{E} = \frac{1}{r} \left( \frac{P_\phi}{r^2} - q \frac{r B}{2} \right) \left( \frac{P_\phi}{r^2} - q \frac{r B}{2} + q r B \right) c^2 \quad \dot{r} = \dot{r} = 0
\]

Here, we have used the fact that the relativistic energy of a particle is given by \( E = \gamma m c^2 \).

2) Clearly, both \( P_\phi \) and \( P_z \) are conserved momenta. Furthermore, we have \( P_r = \gamma m \dot{r} \) and \( P_z = \gamma m \dot{z} \), which is exactly what the ordinary linear relativistic momentum components in the \( r \)- and \( z \)-direction should be. On the other hand, we find that \( P_\phi = \gamma m r^2 \dot{\phi} + q \frac{r^2 B}{2} \). The first expression is equal to the \( z \)-component of the angular momentum of the motion.
(\gamma m r \phi \dot{\phi}) is the linear momentum in phi-direction, and the second term depends on the radius and the magnetic field, so \(P_\phi\) is emphatically not an “ordinary momentum”.

3) To have fixed radius, we must require that

\[ P_e = \dot{P_e} = 0 \Rightarrow \left( \frac{P}{r} - q \frac{rB}{2} \right) \left( \frac{P}{r} - q \frac{rB}{2} + qrB \right) = 0 \]

We note (from above) that the first two terms in each of the two parentheses form the ordinary linear momentum in phi-direction. Therefore, this condition can be fulfilled if either there is no motion in phi, either (first bracket is zero) or if the momentum in phi-direction (the tangential momentum) is equal to \(-qrB\). Of course, the ordinary momentum in z-direction can be anything – it will always be a constant as shown above, and hence the motion is with constant velocity along z and constant angular velocity. For the latter, we can plug in \(\dot{\phi} = \frac{1}{\gamma m} \left( \frac{P}{r} - q \frac{rB}{2} \right) \Rightarrow \frac{1}{\gamma m} \left( -qrB \right) = -\frac{q}{\gamma m} B\) and hence the time for a full circle is \( T = \frac{2\pi}{|\dot{\phi}|} = \frac{2\pi \gamma m}{qB} \). Note that in the case of small velocities (non-relativistic case), this is a constant determined only by \(B\) and the charge-to-mass ratio of the particle.