Classical Mechanics - Problem Set 2 - Solution

Problem 1)

Some appropriate generalized coordinates to use are the distance $r$ of mass 1 from the hole (meaning that mass 2 is a distance $L - r$ below the table) and the angle $\phi$ the string above the table makes with respect to the x-axis. In that case, the total kinetic energy due to both mass points is

$$T = \frac{m_1}{2} (r^2 + \dot{r}^2 \dot{\phi}^2) + \frac{m_2}{2} \dot{r}^2.$$ 

Meanwhile, the potential energy is $V = g m_2 (r - L)$ so that the Lagrangian is

$$L = \frac{m_1}{2} (r^2 + \dot{r}^2 \dot{\phi}^2) + \frac{m_2}{2} \dot{r}^2 - g m_2 (r - L).$$ 

The Euler-Lagrange equations of motion are

$$\frac{d}{dt} \left( m_1 + m_2 \right) \dot{r} = \left( m_1 + m_2 \right) \ddot{r} = \frac{\partial L}{\partial r} = m_1 \dot{r} \dot{\phi}^2 - g m_2,$$

$$\frac{d}{dt} m_1 r^2 \dot{\phi} = 2 m_1 r \dot{r} \dot{\phi} + m_1 r^2 \ddot{\phi} = \frac{\partial L}{\partial \dot{\phi}} = 0.$$

For equilibrium, we have to require that $r = \text{const.}$ and $\dot{\phi} = \text{const.}$ (automatically fulfills 2nd equation) which means that $m_1 \dot{r} \dot{\phi}^2 = g m_2$ or $\dot{\phi} = \sqrt{\frac{g m_2}{r m_1}}$.

Problem 2)

Since there are no external forces, the Lagrangian is simply equal to the kinetic energy, $L = \frac{m}{2} \left( \dot{x}^2 + \dot{y}^2 \right)$. With a single Lagrangian multiplier $\lambda$, we can write the Euler-Lagrange equations following Eq. 2.22 in Goldstein as

$$\frac{d}{dt} m \dot{x} + \lambda 2 \dot{x} = 0 \quad \text{and} \quad \frac{d}{dt} m \dot{y} + \lambda 2 \dot{y} = 0$$

(the second term contains the derivative of the constraint function $g(x, y)$ with respect to $x$ or $y$). Using the ansatz given in the problem, we replace $x$ with $R \cos \phi$ and $y$ with $R \sin \phi$, which yields

$$\frac{d}{dt} m R \left( -\sin \phi \dot{\phi} + 2 \lambda R \cos \phi \right) = 0 \quad \text{and} \quad \frac{d}{dt} m R \cos \phi \dot{\phi} + 2 \lambda R \sin \phi = 0 \quad \Rightarrow \quad m R \left( -\cos \phi \ddot{\phi} - \sin \phi \ddot{\phi} \right) + 2 \lambda R \cos \phi = 0 \quad \text{and} \quad m R \left( -\sin \phi \dot{\phi}^2 + \cos \phi \ddot{\phi} \right) + 2 \lambda R \sin \phi = 0.$$

If we multiply the first equation with $-\sin \phi$ and the second with $\cos \phi$ and add them together, we get $m R \sin^2 \phi \ddot{\phi} + m R \cos^2 \phi \ddot{\phi} = m R \ddot{\phi} = 0$; i.e. $\dot{\phi}$ is constant. Vice versa, if we
multiply the first equation with -cosφ and the second with -sinφ and add them together, we get
\[ mR \cos^2 \phi \dot{\phi}^2 - 2\lambda R \cos \phi + mR \sin^2 \phi \dot{\phi}^2 - 2\lambda R \sin \phi = 0 \Rightarrow 2\lambda = m \dot{\phi}^2. \]
Plugging that back into the original terms in the ELE’s, we see that the magnitude of the force exerted by the string in x-direction is 2\(\lambda x = mx \dot{\phi}^2\) and in y-direction it is 2\(\lambda y = my \dot{\phi}^2\), which are the components of the required centripetal force.

**Problem 3**

We use the generalized coordinates \((\mathbf{R}) = (X,Y,Z)\) and \((\mathbf{r}) = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta)\) which can be used to describe the normal Cartesian coordinates of the two masses \(m_1\) and \(m_2\) like follows:

\[
\begin{align*}
\hat{r}_1 &= \left( x_1, y_1, z_1 \right) = \left( X + \frac{m_2}{m_1 + m_2} r \sin \theta \cos \phi, Y + \frac{m_2}{m_1 + m_2} r \sin \theta \sin \phi, Z + \frac{m_2}{m_1 + m_2} r \cos \theta \right) \\
\hat{r}_2 &= \left( x_2, y_2, z_2 \right) = \left( X - \frac{m_1}{m_1 + m_2} r \sin \theta \cos \phi, Y - \frac{m_1}{m_1 + m_2} r \sin \theta \sin \phi, Z - \frac{m_1}{m_1 + m_2} r \cos \theta \right)
\end{align*}
\]

Calculating the time-derivative for these expressions yields

\[
\frac{d}{dt}(\hat{r}_1) = \left( \dot{x}_1, \dot{y}_1, \dot{z}_1 \right) = \left( \dot{X} + \frac{m_2}{m_1 + m_2} \dot{r}, \dot{Y} + \frac{m_2}{m_1 + m_2} \dot{r}, \dot{Z} + \frac{m_2}{m_1 + m_2} \dot{r} \right); \quad \frac{d}{dt}(\hat{r}_2) = \left( \dot{X} - \frac{m_1}{m_1 + m_2} \dot{r}, \dot{Y} - \frac{m_1}{m_1 + m_2} \dot{r}, \dot{Z} - \frac{m_1}{m_1 + m_2} \dot{r} \right)
\]

After some elementary algebra we find for the kinetic energy

\[
T = \frac{m_1}{2} \dot{X}^2 + \frac{m_2}{2} \dot{r}^2 + \frac{\mu}{2} \dot{r}^2 = \frac{M}{2} \left( \dot{X}^2 + \dot{Y}^2 + \dot{Z}^2 \right) + \frac{\mu}{2} \left( \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right)
\]

Meanwhile, the potential energy is the sum of the two individual potential energies of the two masses in the external field plus the potential energy due to their mutual attraction:

\[
V = m_1 g z_1 + m_2 g z_2 - \frac{G m_1 m_2}{r} = M g Z - G \frac{\mu M}{r} \quad \text{and} \quad L = T - V
\]

(I’ve used the fact that the potential energy BETWEEN the masses is most easily expressed in terms of their relative distance \(r\), which is precisely the reason why this particular set of generalized coordinates is most appropriate for this problem).

The momenta are

\[
\begin{align*}
p_x &= \frac{\partial L}{\partial \dot{X}} = M \dot{X} \\
p_y &= \frac{\partial L}{\partial \dot{Y}} = M \dot{Y} \\
p_z &= \frac{\partial L}{\partial \dot{Z}} = M \dot{Z} \\
p_r &= \frac{\partial L}{\partial \dot{r}} = \mu \dot{r} \\
p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} \\
p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = \mu r^2 \sin^2 \theta \dot{\phi}
\end{align*}
\]
Since the variables \(X, Y\) and \(\phi\) are cyclic (don’t occur in the Lagrangian), the corresponding momenta must all be conserved. (This is obvious since the problem is invariant under translations in \(X\) and \(Y\) and rotations around the \(Z\)-axis.)

Finally, the 6 Lagrange equations read
\[
\begin{align*}
\dot{p}_x &= \mu r = -G \frac{\mu M}{r^2} + \mu \left(r \dot{\theta}^2 + r \sin^2 \theta \dot{\phi}^2\right) ; \\
\dot{p}_\theta &= \mu r^2 \ddot{\theta} + 2 \mu r \dot{r} \dot{\theta} = 0 + \mu r^2 \sin \theta \cos \theta \dot{\phi}^2 ; \\
\dot{p}_\phi &= \mu r^2 \sin^2 \theta \ddot{\phi} + 2 \mu r^2 \sin \theta \cos \theta \dot{\phi} + 2 \mu r \sin^2 \theta \dot{\phi} = 0
\end{align*}
\]

**Problem 4**

The generalized coordinate in this problem is simply \(s = h - z\) (\(z\) is the Cartesian coordinate that measures height above ground zero). Obviously,
\[
T = \frac{m}{2} \dot{s}^2 = \frac{m}{2} \left( gt + A \frac{\pi}{T} \cos \left( \frac{\pi t}{T} \right) \right)^2 = \frac{m}{2} \left( g^2 t^2 + A \frac{\pi^2}{T^2} \cos^2 \left( \frac{\pi t}{T} \right) + 2 g t A \frac{\pi}{T} \cos \left( \frac{\pi t}{T} \right) \right)
\]
while the potential energy is \(V = mgz = mgh - mgs\). This yields for the Lagrangian
\[
L = T - V = \frac{m}{2} \left( g^2 t^2 + A \frac{\pi^2}{T^2} \cos^2 \left( \frac{\pi t}{T} \right) + 2 g t A \frac{\pi}{T} \cos \left( \frac{\pi t}{T} \right) \right) - mgh + \frac{m}{2} g^2 t^2 + mgA \sin \left( \frac{\pi t}{T} \right)
\]
\[
= mg^2 t^2 - mgh + mgA \left( \frac{\pi}{T} t \cos \left( \frac{\pi t}{T} \right) + \sin \left( \frac{\pi t}{T} \right) \right) + \frac{m}{2} A \frac{\pi^2}{T^2} \cos^2 \left( \frac{\pi t}{T} \right)
\]

This result has to be inserted in the integral \(\int_{t=0}^{T} L \, dt\). Note that the first bracket above is simply the total differential with respect to time of \(t \sin \left( \frac{\pi t}{T} \right)\), so the integral becomes
\[
\int_{t=0}^{T} L \, dt = mg^2 \frac{T^3}{3} - mghT + mgAt \sin \left( \frac{\pi t}{T} \right) \bigg|_{t=0}^{t=T} + \frac{m}{2} A \frac{\pi^2}{T^2} \frac{T}{2} = mg^2 \frac{T^3}{3} - mghT + \frac{m}{4} A \frac{\pi^2}{T}
\]

Thus, it is obvious that this integral is minimized if \(A=0\) (all other values of \(A\) will yield a positive additive term).