Classical Mechanics - Problem Set 2

Problem 1)

Two particles of mass \(m_1\) and \(m_2\) are connected by a string of length \(L\) passing through a small hole in a smooth table so that \(m_1\) is situated on the table and \(m_2\) hangs suspended in a gravitational field with acceleration \(g\). Assuming that \(m_1\) moves without friction in the x-y plane of the table, and \(m_2\) moves only vertically along the z-axis parallel to \(g\), write down the Lagrange equations in terms of generalized coordinates of the system. Can this system be in equilibrium?
Assume that neither \(m_1\) nor \(m_2\) passes through the hole.

Problem 2)

A mass point (mass \(m\)) is moving in the x-y plane. There are no external forces present, but the mass is tethered to the origin \((x = y = 0)\) by a string of fixed length \(R\). This constraint can be expressed by requiring that the function \(g(x,y) = 0\), where \(g(x,y) = x^2 + y^2 - R^2\). Using Lagrangian multipliers, write down the Euler-Lagrange equations of motion for the variables \(x\) and \(y\). A solution can be found by making the ansatz \(x = R \cos \phi\) and \(y = R \sin \phi\) for some parameter \(\phi(t)\) (of course we know what that is, but the point is that we’re not supposed to start out with \(\phi\) as generalized coordinate, but only use it at this point). Obviously, this solution “automatically” fulfills the constraint. Show that you can now solve the set of two equations for \(x\) and \(y\) both for \(\phi(t)\) and for the Lagrangian multiplier. From your solution, extract the x- and y-components of the force of constraint exerted by the string on the mass point.
Problem 3)

Two masses $m_1$ and $m_2$ move under their mutual gravitational attraction in a uniform external gravitational field whose acceleration is $g$. Choose as coordinates the Cartesian coordinates $X$, $Y$, $Z$ of the center of mass vector $\mathbf{R}$ (taking $Z$ in the direction of $g$) and the spherical coordinates $r$, $\theta$ and $\phi$ that define the relative vector $\mathbf{r}$ from $m_1$ and $m_2$ (see HW Problem Set 1 for the definition of $\mathbf{r}$). Write down the Lagrangian in terms of these 6 coordinates. Calculate all 6 generalized momenta. Which of these 6 momenta is conserved? Write down the 6 Euler-Lagrange equations of motion.

Problem 4)

A body in a gravitational field (acceleration $g$) is released from a height $h$ and after a time $T$ it strikes the ground, $T = \sqrt{2h/g}$. The equation for the distance of fall $s$ after a time $t$ is of course $s = \frac{1}{2}gt^2$, but assume someone (I) proposes as an alternative the more general form $s = \frac{1}{2}gt^2 + A \sin(\pi t/T)$, where $A$ is an arbitrary constant of dimension length. Note that this gives the same answer for $t = 0$ and $t = T$, meaning the total path $h$ between release and ground contact is traveled in the same time $T$. By calculating the Lagrange function for this proposed trajectory (with $s$ as the single generalized coordinate), find out which value of $A$ yields a minimum for the integral in Hamilton's principle:

$$\int_{t=0}^{t=T} L(s, \dot{s}, t) \, dt.$$

(Note that you have to express $L$ as a function of $t$ first before calculating the integral).