Problem 1)

We can tackle this problem using the energy method. Clearly, here \( V(x) = \frac{k}{2} x^2 + \frac{a}{2x^2} \) so we can write the “energy function” \( h = \frac{m}{2} x^2 + \frac{k}{2} x^2 + \frac{a}{2x^2} = E \). Since the Lagrangian (which I haven’t bothered to write down) does not depend on time, it follows that \( E \) must be a constant of motion. The equilibrium points are given by the extrema of the potential, i.e. for \( F = 0 \) and hence \( x_0 = \pm \sqrt{\frac{a}{k}} \) (two equilibrium points). The second derivative of the potential at these points is \( V'' = k + 3a/x_0^4 = 4k \) which is positive, so the potential has a minimum at each equilibrium point and the equilibrium is stable.

In the usual Taylor expansion, the potential in the vicinity of the equilibrium point can be written as
\[
V(x) = V(x_0) + V'(x_0)(x - x_0) + \frac{1}{2}V''(x_0)(x - x_0)^2 + \cdots = V(x_0) + \frac{1}{2}(4k)(x - x_0)^2
\]
which means that for small deviations from \( x_0 \), the particle will make harmonic oscillations with frequency \( \omega = \sqrt{\frac{4k}{m}} \Rightarrow T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{4k}} = \pi \sqrt{\frac{m}{k}} \).

For larger excursions (larger \( E \)), we can use the fact that we can formally integrate
\[
T = \sqrt{\frac{m}{2}} \int_{x_{\text{min}}}^{x_{\text{max}}} \frac{dx}{\sqrt{E - \frac{k}{2} x^2 - \frac{a}{2x^2}}} = \sqrt{\frac{m}{2}} \int_{x_{\text{min}}}^{x_{\text{max}}} \frac{dx^2}{2 \sqrt{E x^2 - \frac{k}{2} x^4 - \frac{a}{2}}}
\]
where \( x_{\text{min}} \) and \( x_{\text{max}} \) are solutions to the equation \( E = \frac{k}{2} x^2 + \frac{a}{2x^2} \Rightarrow kx^4 - 2Ex^2 + a = 0 \), namely \( x_{\text{min/max}}^2 = \frac{E}{k} \pm \sqrt{\left(\frac{E}{k}\right)^2 - \frac{a}{k}} \). To show that the integral is independent of \( E \), we need to first replace \( x^2 \) with a variable that translates the two limits to terms independent of \( E \). A “straightforward” choice would be the variable transformation
\[ x^2 = \frac{E}{k} + \xi \sqrt{\left(\frac{E}{k}\right)^2 - \frac{a}{k}} \quad \text{where } \xi \text{ runs from } -1 \text{ to } +1. \] Clearly, \( dx^2 = \sqrt{\left(\frac{E}{k}\right)^2 - \frac{a}{k}} \, d\xi \), and the integral becomes

\[
T = \frac{\sqrt{m}}{2} \int_{-1}^{1} \frac{d\xi}{\sqrt{2k \sqrt{\left(\frac{E}{k}\right)^2 - \frac{a}{k}}}}.
\]

Some tedious but elementary algebra reveals that the denominator is equal to

\[
2 \sqrt{E} \left( \frac{E + \xi \sqrt{\left(\frac{E}{k}\right)^2 - \frac{a}{k}}}{k} \right) - \frac{k}{2} \left( \frac{E + \xi \sqrt{\left(\frac{E}{k}\right)^2 - \frac{a}{k}}}{k} \right)^2 - \frac{a}{2} = 2 \sqrt{\frac{E^2}{2k} \left(1 - \xi^2\right) - \frac{a}{2} \left(1 - \xi^2\right)} = \sqrt{2k} \sqrt{\frac{E^2}{k^2} - \frac{a}{k} \sqrt{1 - \xi^2}}
\]

After canceling with the numerator, the integral simplifies to

\[
T = \frac{\sqrt{m}}{k} \int_{-1}^{1} \frac{d\xi}{\sqrt{\left(1 - \xi^2\right)}} \quad \text{⇒} \quad T = \int_{-1}^{1} \frac{d\xi}{\sqrt{\left(1 - \xi^2\right)}}.
\]

Obviously, the integral doesn’t depend anymore on \( E \) (or, for that matter, any other constant of the problem), which is what we needed to show. (It can be easily integrated to

\[
\int_{-1}^{1} \frac{d\xi}{\sqrt{\left(1 - \xi^2\right)}} = \left[ \arcsin(\xi) \right]_{-1}^{1} = \pi, \ q.e.d.
\]

The alternative route to this result would be to argue that, formally, this potential is exactly equal to the pseudopotential “\( V_r(r) \)” that appears in the “energy solution” for the 2-D harmonic oscillator, with \( a = \frac{P_r^2}{m} \). As mentioned in lecture, it is straightforward to solve that problem in ordinary Cartesian coordinates, with the result that the mass point follows an elliptical orbit with a period of 2 times the result for \( T \) above, independent of the initial conditions (and therefore in particular independent of \( E \)). Since every orbit sees two full oscillations from minimum \( r \) to maximum \( r \) and back (for each half of the ellipse), the result above can be inferred directly.
Problem 2)

For circular motion, the major half-axis $a$ is simply the radius of the circle, $R$. If we have a two-particle system, $r$ plays the role of their separation (relative coordinate - each particle is rotating around the common center of mass; see previous HW problems). Using our result from the lecture (or any of the relevant equations in the book), we have

$$\tau = 2\pi \sqrt{\frac{\mu}{k} \frac{R^{3/2}}{GM}}$$

where $M = m_1 + m_2$ is the total mass of the system. When the system gets suddenly stopped, the two masses will fall towards their common center of mass along a straight line. A straight line corresponds to the limit of an “ellipse” with minor half axis $b = 0$ and $e = 1$, which is equivalent with $p_0 = 0$ and therefore $\phi = C$.

Since $r$ has its maximum value at the moment when the two masses get stopped, we could say that the mass $\mu$ in the equivalent 1-particle problem is at its aphelion, which is $2a = R$ from the origin (which in turn in this case is equivalent with the perihelion). So we have $a = R/2$. Furthermore, the time until the collision corresponds to only ½ of a full “revolution” from aphelion to aphelion. So we get for this time

$$t_{coll} = \frac{T}{2} = \frac{2\pi}{2} \sqrt{\frac{(R/2)^3}{GM}} = \frac{2\pi}{4\sqrt{2}} \sqrt{\frac{R^3}{GM}} = \frac{1}{4\sqrt{2}} \tau$$

q.e.d.

The more “brute force” method starts with the law of energy conservation: at the moment both masses stop, their total energy is equal to their gravitational energy, $E = -\frac{GM\mu}{R}$. As they fall to each other, energy conservation says that

$$\frac{\mu}{2} r^2 - \frac{GM\mu}{r} = -\frac{GM\mu}{R} \Rightarrow r = -\sqrt{\frac{2GM}{R}} \sqrt{\frac{R-r}{r}}$$

(the “-“ sign is due to the fact that the two masses move closer to each other, reducing their separation $r$). Using the usual “trick”, we find

$$T = -\sqrt{\frac{R}{2GM}} \int_0^R \left( \frac{R-r}{r} \right)^{-1/2} dr = \frac{\sqrt{R}}{2GM} \int_0^R \frac{r}{R-r} dr.$$  Using the substitution $r = R \sin^2 \xi$, we can evaluate the integral as

$$\int_0^R \frac{1}{\sqrt{R-r}} dr = \int_0^{\pi/2} \frac{1}{\sqrt{R-R \sin^2 \xi}} \sin \xi R 2 \sin \xi \cos \xi d\xi = 2R \int_0^{\pi/2} \sin^2 \xi d\xi = R \int_0^{\pi/2} (1 - \cos(2\xi)) d\xi = R \left[ \frac{\xi}{2} + \frac{1}{2} \sin(2\xi) \right]_0^{\pi/2} = \frac{\pi R}{2}$$

$$\int_0^R (1 - \cos(2\xi)) d\xi = R \left[ \frac{\xi}{2} + \frac{1}{2} \sin(2\xi) \right]_0^{\pi/2} = \frac{\pi R}{2}$$
Plugging it in yields \[ T = \frac{\frac{R}{\sqrt{2GM}}}{\frac{\pi R}{2}} = \frac{\frac{R^3}{GM}}{\frac{2\pi}{4\sqrt{2}}} = \frac{\tau}{4\sqrt{2}}. \]

Problem 3)

From simple energy conservation in Cartesian coordinates ( \( E = \frac{m v^2}{2} - \frac{GMm}{r} \)), we can immediately see that \( v \) is maximal when \( r \) has a minimum and vice versa. That means that the velocity is maximal at the perihelion, \( \phi = 0 \), and minimal at the aphelion, \( \phi = \pi \). In both cases, the velocity vector is perpendicular to the radius vector, which implies that \( v = \frac{\rho \phi}{mr} \). We use the expression \( r = \frac{a(1 - e^2)}{1 + e \cos \phi} \) and find that

\[
\frac{v_{\text{max}}}{v_{\text{min}}} = \frac{\frac{P_\phi}{m(1 - e^2)}}{\frac{P_\phi}{m(1 - e)}} = \frac{1 + e}{1 - e}.
\]

The ratio between these two velocities is simply \( \frac{v_{\text{max}}}{v_{\text{min}}} = \frac{1 + e}{1 - e} \). This is a variation of 3.4% for planet earth, quite noticeable!