Classical Mechanics - Problem Set 3 - Solution

Problem 1)

We can tackle this problem using the energy method. Clearly, here \( V(x) = \frac{k}{2} x^2 + \frac{a}{2x^2} \) so we can write the “energy function” \( h = \frac{m}{2} x^2 + \frac{k}{2} x^2 + \frac{a}{2x^2} = E \). Since the Lagrangian (which I haven’t bothered to write down) does not depend on time, it follows that \( E \) must be a constant of motion. The equilibrium points are given by the extrema of the potential, i.e. for \( F = 0 \) and hence \( x_0 = \pm \sqrt[4]{\frac{a}{k}} \) (two equilibrium points). The second derivative of the potential at these points is \( V'' = k + 3a/x^4 = 4k \) which is positive, so the potential has a minimum at each equilibrium point and the equilibrium is stable.

In the usual Taylor expansion, the potential in the vicinity of the equilibrium point can be written as \( V(x) = V(x_0) + V'(x_0)(x - x_0) + \frac{1}{2} V''(x_0)(x - x_0)^2 + \cdots \approx V(x_0) + \frac{1}{2} (4k)(x - x_0)^2 \) which means that for small deviations from \( x_0 \), the particle will make harmonic oscillations with frequency \( \omega = \sqrt{\frac{4k}{m}} \Rightarrow T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{4k}} = \pi \sqrt{\frac{m}{k}} \).

For larger excursions (larger \( E \)), we can use the fact that we can formally integrate

\[
\frac{T}{2} = \sqrt{\frac{m}{2}} \int_{x_{\min}}^{x_{\max}} \frac{dx}{\sqrt{E - \frac{k}{2} x^2 - \frac{a}{2x^2}}} = \sqrt{\frac{m}{2}} \int_{x_{\min}}^{x_{\max}} \frac{dx}{\sqrt{2Ex^2 - \frac{k}{2} x^4 - \frac{a}{2}}}
\]

where \( x_{\min} \) and \( x_{\max} \) are solutions to the equation \( E = \frac{k}{2} x^2 + \frac{a}{2x^2} \Rightarrow kx^4 - 2Ex^2 + a = 0 \), namely \( x_{\min/\max}^2 = \frac{E}{k} \pm \sqrt{\left( \frac{E}{k} \right)^2 - \frac{a}{k}} \). To show that the integral is independent of \( E \), we need to first replace \( x^2 \) with a variable that translates the two limits to terms independent of \( E \). A “straightforward” choice would be the variable transformation
\[ x^2 = \frac{E}{k} + \xi \sqrt{\left(\frac{E}{k}\right)^2 - \frac{a}{k}} \] where \( \xi \) runs from -1 to +1. Clearly, \( dx^2 = \sqrt{\left(\frac{E}{k}\right)^2 - \frac{a}{k}} d\xi \), and the integral becomes

\[ \frac{T}{2} = \frac{\sqrt{m}}{2} \int_{-1}^{1} \frac{d\xi}{\sqrt{\left(\frac{E}{k}\right)^2 - \frac{a}{k}}} \]

Some tedious but elementary algebra reveals that the denominator is equal to

\[ 2 \sqrt{E \left(\frac{E}{k} + \xi \sqrt{\left(\frac{E}{k}\right)^2 - \frac{a}{k}}\right) - \frac{k}{2} \left(\frac{E}{k} \right)^2 - \frac{a}{k}} - \frac{a}{2} \]

After canceling with the numerator, the integral simplifies to

\[ \frac{T}{2} = \sqrt{\frac{m}{2k}} \int_{-1}^{1} d\xi \Rightarrow T = \sqrt{\frac{m}{k}} \int_{-1}^{1} d\xi \sqrt{\left(1 - \xi^2\right)} \]

Obviously, the integral doesn’t depend anymore on \( E \) (or, for that matter, any other constant of the problem), which is what we needed to show. (It can be easily integrated to

\[ \int_{-1}^{1} \frac{d\xi}{\sqrt{\left(1 - \xi^2\right)}} = \left[\arcsin(\xi)\right]_{-1}^{1} = \pi \text{, q.e.d.} \]

The alternative route to this result would be to argue that, formally, this potential is exactly equal to the pseudopotential “\( V(r) \)” that appears in the “energy solution” for the 2-D harmonic oscillator, with \( a = \frac{P^2}{\epsilon} \). As mentioned in lecture, it is straightforward to solve that problem in ordinary Cartesian coordinates, with the result that the mass point follows an elliptical orbit with a period of 2 times the result for \( T \) above, independent of the initial conditions (and therefore in particular independent of \( E \)). Since every orbit sees two full oscillations from minimum \( r \) to maximum \( r \) and back (for each half of the ellipse), the result above can be inferred directly.
**Problem 2)**

For circular motion, the major half-axis \( a \) is simply the radius of the circle, \( R \). If we have a two-particle system, \( r \) plays the role of their separation (relative coordinate - each particle is rotating around the common center of mass; see previous HW problems). Using our result from the lecture (or any of the relevant equations in the book), we have

\[
\tau = 2\pi \sqrt{\frac{\mu}{k}} \frac{R^{3/2}}{R} = 2\pi \sqrt{\frac{R^3}{GM}} ,
\]

where \( M = m_1 + m_2 \) is the total mass of the system. When the system gets suddenly stopped, the two masses will fall towards their common center of mass along a straight line. A straight line corresponds to the limit of an “ellipse” with minor half axis \( b = 0 \) and \( e = 1 \), which is equivalent with \( p_\theta = 0 \) and therefore \( \phi = C \).

Since \( r \) has its maximum value at the moment when the two masses get stopped, we could say that the mass \( \mu \) in the equivalent 1-particle problem is at its aphelion, which is \( 2a = R \) from the origin (which in turn in this case is equivalent with the perihelion). So we have \( a = R/2 \). Furthermore, the time until the collision corresponds to only \( \frac{1}{2} \) of a full “revolution” from aphelion to aphelion. So we get for this time

\[
t_{\text{coll}} = \frac{T}{2} = \frac{2\pi}{2} \sqrt{\frac{(R/2)^3}{GM}} = \frac{2\pi}{4\sqrt{2}} \sqrt{\frac{R^3}{GM}} = \frac{1}{4\sqrt{2}} \tau ,
\]

q.e.d.

The more “brute force” method starts with the law of energy conservation: at the moment both masses stop, their total energy is equal to their gravitational energy, \( E = -\frac{GM\mu}{R} \). As they fall to each other, energy conservation says that

\[
\frac{\mu}{2} \frac{r^2}{R^2} - \frac{GM\mu}{r} = -\frac{GM\mu}{R} \Rightarrow \frac{r}{R} = \frac{\sqrt{2GM}}{R} \sqrt{\frac{R-r}{r}}
\]

(the “-“ sign is due to the fact that the two masses move closer to each other, reducing their separation \( r \). Using the usual “trick”, we find

\[
T = -\frac{R}{2GM} \int R \left( \frac{R-r}{r} \right)^{-1/2} dr = \frac{R}{2GM} \int \frac{r}{R-r} \sqrt{R-r} dr
\]

Using the substitution \( r = R\sin^2 \xi \), we can evaluate the integral as

\[
\int_0^\pi \frac{\xi}{R-r} dr = \int_0^\pi \frac{R}{R-R\sin^2 \xi} \sin \xi R 2\sin \xi \cos \xi d\xi = 2R \int_0^\pi \sin^2 \xi d\xi = \pi R \\
R \int_0^{\pi/2} (1-\cos 2\xi) d\xi = R \left[ \xi + \frac{1}{2} \sin(2\xi) \right]_0^{\pi/2} = \frac{\pi R}{2}
\]
Plugging it in yields $T = \sqrt{\frac{R \pi R}{2GM}} = \frac{\sqrt{R^3}}{2\sqrt{G\pi}} = \frac{\pi}{4\sqrt{2}}$.

**Problem 3)**

From simple energy conservation in Cartesian coordinates ($E = \frac{m}{2}v^2 - \frac{GMm}{r}$), we can immediately see that $v$ is maximal when $r$ has a minimum and vice versa. That means that the velocity is maximal at the perihelion, $\phi = 0$, and minimal at the aphelion, $\phi = \pi$. In both cases, the velocity vector is perpendicular to the radius vector, which implies that $v = r\phi = \frac{p_\phi}{mr}$. We use the expression $r = \frac{a(1-e^2)}{1+e\cos\phi}$ and find that

$$v_{\text{max}} = \frac{p_\phi}{ma(1-e^2)(1+e)} = \frac{p_\phi}{ma(1-e)}$$

and

$$v_{\text{min}} = \frac{p_\phi}{ma(1-e^2)(1-e)} = \frac{p_\phi}{ma(1+e)}$$

The ratio between these two velocities is simply $\frac{v_{\text{max}}}{v_{\text{min}}} = \frac{1+e}{1-e}$. This is a variation of 3.4% for planet earth, quite noticeable!