Classical Mechanics - Problem Set 4 - Solution

Problem 1)

Solve Goldstein’s problem 6.4 on p.272. Make use of the small-angle approximation 
\[ \sin(\theta) = \theta \text{ and } \cos(\theta) = 1 - \frac{1}{2} \theta^2 \text{ for all angles.} \]

Solution:
The position of the upper mass is given by 
\[ x_1 = l \sin \theta_1, \quad y_1 = -l \cos \theta_1 \Rightarrow \dot{x}_1^2 + \dot{y}_1^2 = l^2 \dot{\theta}_1^2. \]
The position of the lower mass is more complicated:
\[ x_2 = l \sin \theta_1 + l \sin \theta_2, \quad y_2 = -l \cos \theta_1 - l \cos \theta_2 \Rightarrow \dot{x}_2^2 + \dot{y}_2^2 = l^2 \left( (\cos \theta_1 \dot{\theta}_1 + \cos \theta_2 \dot{\theta}_2)^2 + (\sin \theta_1 \dot{\theta}_1 + \sin \theta_2 \dot{\theta}_2)^2 \right) \]
\[ = l^2 \left( \dot{\theta}_1^2 + \dot{\theta}_2^2 + 2 \cos \theta_1 \cos \theta_2 \dot{\theta}_1 \dot{\theta}_2 \right) = l^2 \left( \dot{\theta}_1^2 + \dot{\theta}_2^2 + 2 \cos(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2 \right) \]
The Lagrangian for the double pendulum can be written as
\[ L = \frac{m_1}{2} \dot{\theta}_1^2 + \frac{m_2}{2} \dot{\theta}_2^2 + 2 \cos(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2 + g \left( m_1 \cos \theta_1 + m_2 \left( \cos \theta_1 + \cos \theta_2 \right) \right) \]
\[ = \frac{1}{2} \left[ l^2 (m_1 + m_2) \dot{\theta}_1^2 + l^2 m_2 \dot{\theta}_2^2 + l^2 m_2 \left( 2 - (\theta_1 - \theta_2)^2 \right) \dot{\theta}_1 \dot{\theta}_2 \right] + g l (m_1 + 2 m_2) \]
\[ = \frac{1}{2} \left[ l^2 (m_1 + m_2) \dot{\theta}_1^2 + l^2 m_2 \dot{\theta}_2^2 + g l m_2 \dot{\theta}_2^2 \right] \]

where in the 2nd line I used the small-angle approximation. If we further suppress the 
term proportional to 4 small angles (angular velocities), 
\[ (\theta_1 - \theta_2)^2 \dot{\theta}_1 \dot{\theta}_2, \] we can immediately read off the two matrices introduced in lecture (also ignoring the constant term in 
the potential energy):
\[
\begin{pmatrix}
T \\
\end{pmatrix} = \begin{pmatrix}
l^2 (m_1 + m_2) & l^2 m_2 \\
l^2 m_2 & l^2 m_2 \\
\end{pmatrix};
\begin{pmatrix}
V \\
\end{pmatrix} = \begin{pmatrix}
g l (m_1 + m_2) & 0 \\
0 & g l m_2 \\
\end{pmatrix}.
\]
We have to solve the secular equation
The solutions are:

\[ \lambda_{1,2} = \frac{g}{l} \left[ \left(1 + \frac{m_2}{m_1}\right) \pm \sqrt{\frac{m_2}{m_1} \left(1 + \frac{m_2}{m_1}\right)} \right] = \omega_0 \left[ (1 + r) \pm \sqrt{r(1 + r)} \right] \]

with “obvious” definitions for \( \omega_0 \) and \( r \). We now have to find the solution for the eigenvectors.

For now, we can arbitrary choose \((a_0) = 1\). Then the first row of the eigenvector equation reads, after dividing all matrices by \( l^2 \) and by \( m_1 \):

\[
0 = [v] - \lambda [T] = \begin{bmatrix} g l (m_1 + m_2) - \lambda l^2 (m_1 + m_2) & -\lambda l^2 m_2 \\ -\lambda l^2 m_2 & g l m_2 - \lambda l^2 m_2 \end{bmatrix} = \begin{bmatrix} \frac{g}{l} (m_1 + m_2) - \lambda (m_1 + m_2) \frac{g}{l} m_2 - \lambda m_2 \end{bmatrix} l^4 = \begin{bmatrix} \frac{g}{l^2} (m_1 + m_2) m_2 - \lambda \left( m_1 + m_2 \right) \frac{g}{l} m_2 + \frac{g}{l} \left( m_1 + m_2 \right) m_2 \right] + \lambda^2 \left[ (m_1 + m_2) m_2 - m_2 \right] l^4 = \begin{bmatrix} \frac{g}{l^2} \left( m_1 + m_2 \right) m_2 - 2 \lambda \frac{g}{l} \left( m_1 + m_2 \right) m_2 + \lambda^2 m_2 m_2 \right] l^4 = \begin{bmatrix} \frac{g}{l^2} \left( \frac{m_2}{m_1} \right) - 2 \lambda \frac{g}{l} \left( \frac{m_2}{m_1} \right) + \lambda^2 \right] m_2 l^4 \]

So we have two fundamental solutions (eigenmodes) for the double pendulum:

1) One where both masses are moving in phase, with the same angular velocity, but different amplitudes (the lower one with a larger amplitude than the upper one) and with a frequency \( \omega = \omega_0 \sqrt{\left(1 + r\right) - \sqrt{r(1 + r)}} \) that is typically slower than that of a single pendulum with length \( l \). In particular, for equal masses \( (r = 1) \), the frequency is about 77% of \( \omega_0 \), while for very small mass \( m_2 \), it goes towards \( \omega_0 \).
2) The other one has a much faster frequency, \( \omega = \omega_0 \sqrt{1 + r} + \sqrt{r(1 + r)} \) but the two masses move in opposite direction. In the case when mass 2 is much larger than mass 1, we can see that the lower mass stays nearly still while the upper oscillates back and forth quickly.

In general, if mass 2 is small (hence \( r \) is small), then the two frequencies become similar, while the excursion of the lower bob becomes larger and larger relative to the upper one. Finally, if initially we just move the upper mass from equilibrium, we have an initial state that is a combination of the two eigenmodes. Hence we have to find a linear combination that fits the initial conditions:

\[
\theta_1 = C_1 a_{11} e^{-i\omega_1 t} + C_2 a_{21} e^{-i\omega_2 t}; \quad \theta_2 = C_1 a_{12} e^{-i\omega_1 t} + C_2 a_{22} e^{-i\omega_2 t}
\]

with \( a_{11} = a_{21} = 1 \),

\[
a_{12} = -\frac{m_1}{m_2^2} + 1; \quad a_{22} = \frac{m_1}{m_2} + 1; \quad \omega_1 = \omega_0 \sqrt{1 + r} + \sqrt{r(1 + r)}; \quad \omega_2 = \omega_0 \sqrt{1 + r} - \sqrt{r(1 + r)}.
\]

Since both masses have initial velocity 0, we can choose both constants \( C_1 \) and \( C_2 \) as purely real and replace the exponents with cosines. Plugging in the initial conditions for \( t = 0 \), we find

\[
C_1 + C_2 a_{21} = \theta_1(0) \Rightarrow C_1 = \theta_1(0) - C_2 \quad \text{and} \quad C_1 a_{12} + C_2 a_{22} = \sqrt{\frac{m_1}{m_2} + 1} (C_2 - C_1) = \sqrt{\frac{m_1}{m_2} + 1} (2C_2 - \theta_1(0)) = 0
\]

\[
\Rightarrow \quad C_2 = \frac{\theta_1(0)}{2} = C_1
\]

so the full motion is described by

\[
\theta_1 = \frac{\theta_1(0)}{2} \left( \cos(\omega_1 t) + \cos(\omega_2 t) \right); \quad \theta_2 = \frac{\theta_1(0)}{2} \sqrt{\frac{m_1}{m_2} + 1} (\cos(\omega_1 t) - \cos(\omega_2 t))
\]

It is clear that this fulfills the initial condition for \( t = 0 \). Initially, while the two cosines are in phase, only the upper mass will oscillate back and forth while the lower mass sits still. However, as time passes, the two cosines fall out of phase, the upper mass reduces its amplitude while the lower mass increases its motion. Once the two cosines are 180 degrees out of phase, the situation has reversed – now the upper mass sits (nearly) still while the lower mass moves with an amplitude of \( \theta_1(0) \sqrt{\frac{m_1}{m_2}} + 1 \). This happens after a time \( T \) with \( \omega_1 T = \omega_2 T + \pi \) or \( T = \frac{\pi}{\omega_1 - \omega_2} = \frac{\pi \sqrt{\frac{m_1}{m_2} + 1}}{\omega_0 \left( \sqrt{1 + r} + \sqrt{r(1 + r)} - \sqrt{1 + r} - \sqrt{r(1 + r)} \right)} \),
the “beat time”. Hence we will go back and forth from one mode to another where either one of the two masses makes most of the motion.

**Problem 2)**

Three massless rods are free to move on a common axis at the end of each rod. The other end of each rod is attached to a mass \( m \) (all 3 masses are identical). Rods 1 and 2, 2 and 3 and 1 are connected by torsional springs with torque \( \tau = -\kappa(\theta - \theta_0) \), where \( \theta(t) \) is the instant angle between two adjacent rods, and \( \theta_0 = 2\pi/3 \) is the equilibrium angle for the spring, and \( \kappa \) is a constant. Write the Lagrangian for this system, find the equation of motion for each mass and determine the normal mode frequencies and eigenvectors. What motion does each eigenvector describe?

**Solution:**

The kinetic energy is straightforward to write down in terms of the masses and the length \( l \) of the rods:

\[
T = \frac{ml^2}{2} \left( \dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2 \right)
\]

The potential energy is

\[
V = \frac{\kappa}{2} \left[ (\theta_1 - \theta_2 - \theta_0)^2 + (\theta_2 - \theta_3 - \theta_0)^2 + (\theta_3 - \theta_1 - \theta_0)^2 \right] = \frac{\kappa}{2} \left[ 2\theta_1^2 + 2\theta_2^2 + 2\theta_3^2 - 2\theta_1\theta_2 - 2\theta_2\theta_3 - 2\theta_3\theta_1 + 3\theta_0^2 \right]
\]

Fortunately, all the terms linear in \( \theta_0 \) cancel, and as always, we can ignore any constant offset in the potential energy. Hence, we can write the Lagrangian in the usual (here: 3x3) matrix form (note that here the angles are not necessarily small, but there are no higher order terms to worry about):

\[
L = \frac{ml^2}{2} \left( \dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2 \right) - \frac{\kappa}{2} \left[ 2\theta_1^2 + 2\theta_2^2 + 2\theta_3^2 - 2\theta_1\theta_2 - 2\theta_2\theta_3 - 2\theta_3\theta_1 + 3\theta_0^2 \right] = \frac{1}{2} \left( \ddot{\theta} \right)^T \begin{pmatrix} m\dot{\theta}_1^2 & 0 & 0 \\ 0 & m\dot{\theta}_2^2 & 0 \\ 0 & 0 & m\dot{\theta}_3^2 \end{pmatrix} \left( \dot{\theta} \right) - \frac{1}{2} \left( \dot{\theta} \right)^T \begin{pmatrix} 2\kappa & -\kappa & -\kappa \\ -\kappa & 2\kappa & -\kappa \\ -\kappa & -\kappa & 2\kappa \end{pmatrix} \left( \dot{\theta} \right)
\]
Once again, we have to solve the characteristic equation

\[
0 = \begin{vmatrix}
2\kappa - \lambda m l^2 & -\kappa & -\kappa \\
-\kappa & 2\kappa - \lambda m l^2 & -\kappa \\
-\kappa & -\kappa & 2\kappa - \lambda m l^2 \\
\end{vmatrix} = (2\kappa - \lambda m l^2)^3 - 3(2\kappa - \lambda m l^2)\kappa^2 - 2\kappa^3
\]

\[
= 8\kappa^3 - 12\kappa^2 \lambda m l^2 + 6\kappa^2 m^2 l^4 - \lambda^3 m l^6 - 6\kappa^3 + 3\kappa^2 \lambda m l^2 - 2\kappa^3 = -9\kappa^2 \lambda m l^2 + 6\kappa^2 m^2 l^4 - \lambda^3 m l^6
\]

Clearly, ONE possible solution is \( \lambda = 0! \) Inspection of \((v)\) shows that the corresponding eigenvector is simply \( \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \). This corresponds to a motion where all three masses move in sync and furthermore do not oscillate but keep going at constant angular velocity – which is of course a trivial solution (all three particles have a constant distance of \(2\pi/3\) from each other, with all springs relaxed, and the whole arrangement rotating as if a solid. To find the other non-zero solutions, we can divide by \(\lambda m^3 l^6\) and find

\[
\lambda^2 - 6\frac{\kappa}{m l^2} \lambda = -\frac{9\kappa^2}{m^2 l^4} \Rightarrow \lambda_{2,3} = \frac{3\kappa}{m l^2} \] (the 2\(^{nd}\) and 3\(^{rd}\) eigenvalues are degenerate). It follows that

\[
\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\kappa & -\kappa & -\kappa \\ -\kappa & -\kappa & -\kappa \\ -\kappa & -\kappa & -\kappa \end{pmatrix} .
\]

We have to find two possible solutions to

\[
\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0. \]

Again we can arbitrarily choose \( a_1 = 1 \). The simplest choice is to pick either \( a_2 \) or \( a_3 \) equal to -1 and the other equal to zero. This corresponds to a solution where one of the 3 masses remains at rest while the other two move back and forth 180 degree out of phase relative to each other, with frequency \( \omega = \sqrt{\frac{3\kappa}{m l^2}} \). Of course, the most general solution involves a superposition of these three eigenmodes – an overall rotation and an oscillation of 1 vs. 2 or 1 vs. 3.