Classical Mechanics - Problem Set 5 (Due Thursday, February 27)

Problem 1)

Assume you are given a rotational matrix \( R \) that transforms the components of the vector \( r \) in the unprimed coordinate system, \((r) = (x, y, z)\), into its components in the primed (rotated) coordinate system, \((r)' = (x', y', z') = (R)(r)\) (passive rotation).

Show explicitly (using known properties of derivatives and rotational matrices) that the gradient
\[
\mathbf{\nabla} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)
\]
transforms like a vector under this rotation,
\[
\mathbf{\nabla}' = \left( \frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'} \right) = R \cdot \mathbf{\nabla}
\]
where the components of \( \mathbf{\nabla} \) in the primed coordinate system are simply the derivatives with respect to the primed coordinates \( x', y', z' \).

(If you find the notion of a “del operator” without anything to take the derivative of unfamiliar, just “multiply” it with any scalar function \( f(r) \), i.e., take the gradient of such a function).

Problem 2)

Check Goldstein’s Eq. (4.46), pg. 153, by explicitly multiplying the three “Euler matrices” BCD. Show your intermediate steps.

Problem 3)

An infinitesimal rotation around the z-axis by an angle \( d\phi \) can be described by the rotational matrix \( R(d\phi) = 1 + M_3 d\phi \) (see Goldstein pg. 171). Show that for a finite rotation around the same axis by an angle \( \phi \), one can write the rotational matrix as
\[
R(\phi) = \exp(M_3 \phi)
\]
where the “exponent of a matrix” is simply given by the usual Taylor expansion of the exp function. Note: your proof doesn’t have to be rigorous – I’ll give you full credit if you calculate the first few terms in the Taylor expansion and then explain how the full series will lead to the correct result.