Classical Mechanics - Problem Set 7 - Solution

Problem 1)

We start by quoting the Lagrangian for problem 3, Problem Set 2:
\[ L = T - V = \frac{M}{2} \left( \dot{X}^2 + \dot{Y}^2 + \dot{Z}^2 \right) + \frac{\mu}{2} \left( \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \varphi \dot{\varphi}^2 \right) - MgZ + G \frac{\mu M}{r} \]

We also calculated the generalized momenta back then:
\[ p_x = \frac{\partial T}{\partial \dot{X}} = MX; \quad p_y = \frac{\partial T}{\partial \dot{Y}} = MY; \quad p_z = \frac{\partial T}{\partial \dot{Z}} = MZ \]
\[ p_r = \frac{\partial T}{\partial r} = \mu r; \quad p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \mu r^2 \dot{\theta}; \quad p_\varphi = \frac{\partial T}{\partial \dot{\varphi}} = \mu r^2 \sin^2 \theta \dot{\varphi} \]

Obviously, the Hamiltonian here is simply the total energy: \( H = T + V \). Expressing all velocities in terms of the generalized momenta, I get
\[ H = \frac{p_x^2 + p_y^2 + p_z^2}{2M} + \frac{p_r^2}{2\mu} + \frac{p_\theta^2}{2\mu r^2} + \frac{p_\varphi^2}{2\mu r^2 \sin^2 \theta} + MgZ - G \frac{\mu M}{r} . \]

Clearly, in this case total energy is conserved (no explicit t-dependence in \( H \)).

Hamilton’s equations yield the following expressions (the first 6 are already contained in the definition of the momenta above):
\[ \dot{X} = \frac{\partial H}{\partial p_x} = \frac{p_x}{M}; \quad \dot{Y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{M}; \quad \dot{Z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{M} ; \]
\[ \dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{\mu}; \quad \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{\mu r^2}; \quad \dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{\mu r^2 \sin^2 \theta} ; \]
\[ \dot{p}_x = -\frac{\partial H}{\partial X} = 0; \quad \dot{p}_y = -\frac{\partial H}{\partial Y} = 0; \quad \dot{p}_z = -\frac{\partial H}{\partial Z} = -Mg; \]
\[ \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_r^2}{\mu r^3} + \frac{p_\varphi^2}{\mu r^3 \sin^2 \theta} - G \frac{\mu M}{r^2} ; \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\theta^2 \cos \theta}{\mu r^2 \sin \theta} ; \quad \dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = 0 . \]

The solutions for the 3 CoM momenta are
\[ p_x = P_{x0} = const., \quad p_r = P_{r0} = const., \quad p_z = P_{z0} - Mgt \]. These allow us to integrate the equations for the 3 CoM velocity components:
\[ X(t) = X_0 + \frac{P_{x0}}{M} t; \quad Y(t) = Y_0 + \frac{P_{r0}}{M} t; \quad Z(t) = Z_0 + \frac{P_{z0}}{M} t - \frac{1}{2} gt^2 \]

Problem 2)

From the lecture (or the book) we know that \( L = \frac{m}{2} \dot{\vec{v}}^2 + qAx \dot{y} = \frac{m}{2} \dot{\vec{v}}^2 + qxB_0 \dot{y} \) (the electric potential term drops). The generalized momenta are
\[ p_x = \frac{\partial L}{\partial \dot{X}} = m \dot{X}, \quad p_y = \frac{\partial L}{\partial \dot{Y}} = m \dot{Y} + qxB_0 \]

and $p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$. The Hamiltonian is therefore

$$H = \dot{v} \cdot \vec{p} - L = \frac{m}{2} \dot{v}^2 + qxB_0 \dot{y} - qxB_0 \dot{y} = \frac{m}{2} \dot{v}^2.$$  

Apparently, the value of the Hamiltonian is just the kinetic energy, which is indeed the total energy of the system, since magnetic fields don’t do any work. However, we have to express the Hamiltonian in terms of the canonical momenta: $H = \frac{1}{2m} \left( p_x^2 + (p_y - qxB_0)^2 + p_z^2 \right)$. Since it doesn’t depend on time explicitly, the value of the Hamiltonian (the energy) is conserved. The Hamilton Equations of motion read as follows:

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}; \quad \dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y - qxB_0}{m}; \quad \dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m};$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = \frac{p_x - qxB_0}{m} qB_0 = \dot{y} qB_0; \quad \dot{p}_y = -\frac{\partial H}{\partial y} = 0; \quad \dot{p}_z = -\frac{\partial H}{\partial z} = 0.$$

Of course, the first line brings nothing new. By inserting the velocities in the second line, we find $m\ddot{x} = \frac{p_y qB_0}{m} - \frac{q^2 B_0^2}{m} x \Rightarrow \ddot{x} = \frac{q^2 B_0^2}{m^2} \left( \frac{p_y}{qB_0} - x \right)$. Since $P_y$ is a constant, this is just the harmonic oscillator equation with an offset in $x$, so the solution is $x(t) = \frac{p_y}{qB_0} + A \sin(\omega t + \varphi)$ with $\omega = \frac{qB_0}{m}$. The 2nd expression in the 1st line yields $\dot{y} = \frac{p_y}{m} - \omega x = -\omega A \sin(\omega t + \varphi) \Rightarrow y(t) = C + A \cos(\omega t + \varphi)$. Together, the two expressions for $x$ and $y$ describe a circular motion with angular velocity $\omega$ around a center point given by $(x_c, y_c) = \left( \frac{p_y}{qB_0}, C \right)$. The integration constants $P_y, A, C, \varphi$ are given by the initial conditions. Finally, using the last terms in each line, we get $\dot{z} = \frac{p_z}{m} = \text{const.} \Rightarrow z(t) = z_0 + \frac{p_z}{m} t$. Of course, given that $T$ is conserved, the magnitude of the linear velocity is also constant. The resulting motion therefore is a circular motion in $x$ and $y$, superimposed with a linear motion along $z$ – a helix. This system is a (rare) example where Hamilton’s equations of motion give a more direct solution without any need for additional arguments (e.g., Forces that are perpendicular to the velocity in the x-y plane etc.).