Classical Mechanics - Problem Set 8 - Solution

Problem 1)

The proposed transformations are:
\[ Q = q \cos \alpha - p \sin \alpha; \quad P = q \sin \alpha + p \cos \alpha \]

To show that they are canonical transformations, we have to show that they fulfill the symplectic condition.

To begin, we have to calculate all of the relevant derivatives to get the matrix \( M \):
\[
\begin{align*}
\frac{\partial Q}{\partial q} &= \cos \alpha; \quad \frac{\partial Q}{\partial p} = -\sin \alpha; \quad \frac{\partial P}{\partial q} = \sin \alpha; \quad \frac{\partial P}{\partial p} = \cos \alpha
\end{align*}
\]

\[
M = \begin{pmatrix}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha
\end{pmatrix}
\]

Checking the symplectic condition:
\[
(M^{-1})^T (J) (M) = \begin{pmatrix}
\cos \alpha & \sin \alpha \\
-\sin \alpha & \cos \alpha
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\]

q.e.d.

To find a generating function of type “\( F_1 \)”, we have to first invert the transformations to express \( p \) and \( P \) in terms of \( q \) and \( Q \):
\[
p = \frac{q \cos \alpha - Q}{\sin \alpha}; \quad P = q \sin \alpha + \frac{q \cos \alpha - Q}{\cos \alpha} = \frac{q - Q \cos \alpha}{\sin \alpha}
\]

Then we integrate the equations 9.14 in the book:
\[
\begin{align*}
\frac{\partial F_1}{\partial q} &= p = \frac{q \cos \alpha - Q}{\sin \alpha} \Rightarrow F_1 = \frac{1}{2} q^2 \cos \alpha - \frac{Q q}{\sin \alpha} + f(Q) \\
\Rightarrow \quad \frac{\partial F_1}{\partial Q} &= -\frac{q}{\sin \alpha} + f'(Q) = -P = \frac{-q}{\sin \alpha} + \frac{Q \cos \alpha}{\sin \alpha} \Rightarrow f(Q) = \frac{1}{2} Q^2 \cos \alpha + C
\end{align*}
\]

Note that this only works because \[ \frac{dp(q,Q)}{dQ} = \frac{dP(q,Q)}{dq} \] as required for the 2nd derivatives of a function \( F_1 \). The constant \( C \) is arbitrary and might as well be chosen to be zero.

Hence, a possible generating function of type “\( F_1 \)” would be the following:
\[
F_1(q,Q) = \frac{1}{2} \left( q^2 + Q^2 \right) \cos \alpha - \frac{q Q}{\sin \alpha}.
\]
(Checking with Eqns. (9.14, we find that indeed

\[
p = \frac{\partial F_1}{\partial q} = q\cos\alpha - \frac{Q\cos\alpha - (q\cos\alpha - p\sin\alpha)}{\sin\alpha}
\]
and

\[
P = -\frac{\partial F_1}{\partial Q} = -\frac{Q\cos\alpha - q}{\sin\alpha} = \frac{q - (q\cos\alpha - p\sin\alpha)\cos\alpha}{\sin\alpha}
\]

\[
= \frac{q\sin^2\alpha + p\sin\alpha\cos\alpha}{\sin\alpha} = q\sin\alpha + p\cos\alpha
\]
as required.)

The form of the Hamiltonian itself is unchanged since \( \frac{\partial F_1}{\partial t} = 0 \). However, we have to replace \( p \) with \( p = -Q\sin\alpha + P\cos\alpha \) and \( q \) with \( q = Q\cos\alpha + P\sin\alpha \), i.e. the inverse of the original transformation equations, wherever they appear in the Hamiltonian.

The given transformation describes the arbitrary mixing of position and momentum variables (not even with the same dimensions!). For the case \( \alpha = 0 \) we have of course the identity transformation (although the generating function “does not work” for this particular case – one would have to use a generating function of type “F₂” instead). For the case \( \alpha = \pi/2 \) we simply identify \( Q \) with \(-p\) and \( P \) with \( q \); the generating function \( F_1 = -qQ \) works in this case (as already shown in lecture).

**Problem 2)**

First we show that the transformation is canonical, by showing that its **INVERSE** fulfills the symplectic condition:

\[
(M) = \begin{pmatrix}
\frac{\partial x}{\partial Q_1} & \frac{\partial x}{\partial Q_2} & \frac{\partial x}{\partial P_1} & \frac{\partial x}{\partial P_2} \\
\frac{\partial y}{\partial Q_1} & \frac{\partial y}{\partial Q_2} & \frac{\partial y}{\partial P_1} & \frac{\partial y}{\partial P_2} \\
\frac{\partial p_x}{\partial Q_1} & \frac{\partial p_x}{\partial Q_2} & \frac{\partial p_x}{\partial P_1} & \frac{\partial p_x}{\partial P_2} \\
\frac{\partial p_y}{\partial Q_1} & \frac{\partial p_y}{\partial Q_2} & \frac{\partial p_y}{\partial P_1} & \frac{\partial p_y}{\partial P_2}
\end{pmatrix} = \begin{pmatrix}
\frac{\sqrt{2P_1} \cos Q_1}{\alpha} & 0 & \frac{\sin Q_1}{\alpha\sqrt{2P_1}} & \frac{1}{\alpha} \\
0 & \frac{\sqrt{2P_1} \sin Q_1}{\alpha} & \frac{\cos Q_1}{\alpha\sqrt{2P_1}} & 0 \\
0 & 0 & -\frac{\alpha \cos Q_1}{2\sqrt{2P_1}} & 0 \\
0 & 0 & \frac{\alpha \sin Q_1}{2\sqrt{2P_1}} & \frac{1}{\alpha}
\end{pmatrix}
\]
\[
\begin{pmatrix}
\sqrt{2}P_1 \cos Q_i & -\sqrt{2}P_1 \sin Q_i & -\alpha \sqrt{2}P_1 \sin Q_i & -\alpha \sqrt{2}P_1 \cos Q_i \\
\frac{\alpha}{2} & 0 & -\frac{\alpha}{2} & 0 \\
0 & 1 & -\frac{\alpha}{\sqrt{2} P_1} & 0 \\
\sin Q_i & \cos Q_i & \alpha \cos Q_i & -\alpha \sin Q_i \\
\alpha \sqrt{2} P_1 & \alpha \sqrt{2} P_1 & 2 \sqrt{2} P_1 & 2 \sqrt{2} P_1 \\
\frac{1}{\alpha} & 0 & 0 & \alpha \\
\end{pmatrix}
\begin{pmatrix}
\sqrt{2} P_1 \cos Q_i & 0 & \sin Q_i & \frac{1}{\alpha} \\
\alpha & 0 & \frac{1}{\alpha} & 0 \\
\frac{\alpha}{2} & 0 & \frac{1}{\alpha} & 0 \\
\frac{\alpha}{2} \sin Q_i & 0 & \frac{\alpha}{2} \cos Q_i & 0 \\
\frac{\alpha}{2} \cos Q_i & \frac{\alpha}{2} \cos Q_i & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} - \sin^2 Q_i - \cos^2 Q_i & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} \frac{1}{2} & 0 \\
\end{pmatrix}
\]

q.e.d.

The Hamiltonian in ordinary coordinates for this problem is

\[
H = \frac{(\vec{p} - q A)^2}{2m} = \frac{(p_x + \frac{\gamma}{2} q B_0)^2}{2m} + \frac{(p_y - \frac{\gamma}{2} q B_0)^2}{2m}.
\]

Plugging in the equations given in the problem yields
This Hamiltonian is obviously independent of both $Q_1$ and $Q_2$ (both are cyclic) and therefore we can immediately infer a solution:

\[ \dot{P}_1 = \frac{mE}{qB_0} = \text{const.} ; \quad \dot{P}_2 = \text{const.} \]

\[ \dot{Q}_1 = \frac{\partial H}{\partial \dot{P}_1} = \frac{qB_0}{m} = \omega ; \quad Q_1 = \omega t + \varphi_0 ; \quad \dot{Q}_2 = \frac{\partial H}{\partial \dot{P}_2} = 0 \quad \Rightarrow \quad Q_2 = \text{const.} \]

Obviously, the motion is circular with a radius of \( R = \frac{\sqrt{2P_1}}{\alpha} = \frac{\sqrt{2mE}}{\alpha^2} = \frac{mv}{qB_0} \) (\( v \) is the ordinary speed – remember that the value of \( H \) is simply the total energy which here is the kinetic energy alone) and an angular velocity \( \omega = \frac{qB_0}{m} \), around a fixed center located at

\[ (x_0, y_0) = \left( \frac{P_2}{\alpha}, \frac{Q_2}{\alpha} \right). \]