Classical Mechanics - Problem Set 9 - Solution

Problem 1)

As we know, the Hamiltonian for this case is \( H = \frac{(\vec{p} - q\vec{A})^2}{2m} = \frac{(\vec{p} - \frac{q}{2}\vec{B} \times \vec{r})^2}{2m} \) and therefore the velocity vector is given by \( \vec{v} = \frac{\partial H}{\partial \vec{p}} = \frac{\vec{p} - \frac{q}{2}\vec{B} \times \vec{r}}{m} \). Evaluating the Poisson brackets yields:

a)

\[
\begin{align*}
[\nu_x, \nu_y] &= \sum_{i=1}^{3} \left( \frac{\partial \nu_x}{\partial r_i} \frac{\partial \nu_y}{\partial p_i} - \frac{\partial \nu_x}{\partial p_i} \frac{\partial \nu_y}{\partial r_i} \right) = \frac{1}{m^2} \sum_{i=1}^{3} \left( \frac{\partial (p_x - \frac{q}{2}B_yz + \frac{q}{2}B_zy)}{\partial r_i} \frac{\partial (p_y - \frac{q}{2}B_zx + \frac{q}{2}B_xz)}{\partial p_i} \right) \\
&= \frac{1}{m^2} \left[ \frac{q}{2}B_z \cdot 1 - \frac{q}{2}B_y \cdot 0 - \left( -\frac{q}{2}B_z \cdot 1 + \frac{q}{2}B_x \cdot 0 \right) \right] = \frac{q}{m^2} B_z; \\
[\nu_x, \nu_z] &= \frac{q}{m^2} B_y; \\
[\nu_z, \nu_x] &= \frac{q}{m^2} B_x
\end{align*}
\]

where the last two equations follow by permutation and the remaining three Poisson brackets are simply the negative of the listed ones.

b)

\[
\begin{align*}
[r_i, \nu_j] &= \frac{\partial \nu_j}{\partial p_i} - \frac{1}{m} \delta_{ij} ; \\
[p_i, \nu_j] &= -\frac{\partial \nu_j}{\partial r_i} = \frac{q}{2m} \sum_{k=1}^{3} \varepsilon_{jki} B_k
\end{align*}
\]

For the following, we recall that \( \dot{p}_j = -\frac{\partial H}{\partial r_j} = \frac{\vec{p} - q\vec{A}}{m} \frac{\partial q\vec{A}}{\partial r_j} \). From this it follows that

\[
\begin{align*}
[r_i, \dot{p}_j] &= \frac{\partial \dot{p}_j}{\partial p_i} = \frac{q}{m} \frac{\partial A}{\partial r_j} = \frac{q}{2m} \sum_{k=1}^{3} \varepsilon_{ikj} B_k
\end{align*}
\]

\[
\begin{align*}
[p_i, \dot{p}_j] &= -\frac{\partial \dot{p}_j}{\partial r_i} = \frac{q^2}{m} \frac{\partial A}{\partial r_j} \frac{\partial A}{\partial r_j} = \frac{q^2}{4m} \sum_{l} \left( \sum_{k} \varepsilon_{lki} B_k \sum_{m} \varepsilon_{lmj} B_m \right) = \frac{q^2}{4m} \sum_{l, k, m} \varepsilon_{lkj} \varepsilon_{lmj} B_l B_k
\end{align*}
\]

This last expression can be shown, with some algebraic tedium, to equal to \( \frac{q^2}{4m} \left( B^2 \delta_{ij} - B_j B_i \right) \). This result can also be gotten if we observe that \( [p_i, \dot{p}_j] = \frac{\partial^2 H}{\partial r_i \partial r_j} \).

The second derivative is only non-zero for terms that are at least quadratic in the coordi-
The only such term occurring in the Hamiltonian is
\[ \left( -\frac{q}{2} \cdot \vec{B} \times \vec{r} \right)^2 = \frac{d^2}{8m} \vec{B} \cdot \vec{r} \times \left( \vec{B} \times \vec{r} \right). \] Using the properties of the product
\[ A \cdot (\vec{B} \times \vec{C}) = B \cdot (\vec{C} \times \vec{A}), \] we can rewrite this term as \[ \frac{d^2}{8m} \vec{B} \cdot \left( \vec{B} \times \vec{r} \right)^2 \] which, according to the BAC-CAB rule, is equal to \[ \frac{d^2}{8m} \vec{B} \cdot \left( \vec{B} \times \vec{r} \right)^2 - \vec{B} \cdot \left( \vec{B} \times \vec{r} \right) \cdot \vec{B} \cdot \vec{r}. \] Taking the second partial derivatives of the last expression yields the same result as the first method.

**Problem 2)**

a)
\[ [H, [u, v]] = -[u[H, u]] - [v[H, u]] = - \left( u_r \frac{\partial v}{\partial t} - v_r \frac{\partial u}{\partial t} \right) = \left( u_r \frac{\partial v}{\partial t}, v_r \frac{\partial u}{\partial t} \right) = \frac{\partial}{\partial t} [u, v] \quad \text{q.e.d.} \]

The last equality follows because the partial derivative with respect to time commutes with the partial derivatives with respect to the canonical coordinates and momenta inside the Poisson brackets.

b) This follows from induction, using our result from a): If \( F \) and \( H \) are constants of motion, then \[ [H, F] = \frac{\partial F}{\partial t} \] is a constant of motion, too. If the \( n \)-th partial derivative of \( F \) is a constant of motion, then the \( n+1 \)-th one is also: \[ [H, \frac{\partial^n F}{\partial t^n}] = \frac{\partial}{\partial t} \left( \frac{\partial^n F}{\partial t^n} \right) = \frac{\partial^{n+1} F}{\partial t^{n+1}}, \] where the Poisson bracket is a constant of motion following a).

c) Here, \( F = x - \frac{pt}{m} \) is simply the initial position \( x(t=0) \) of the free particle, which is clearly a constant of motion. \( \frac{\partial F}{\partial t} = -\frac{p}{m} \) is also a constant of motion (the negative of the constant velocity). \( H = \frac{p^2}{2m} \) of course is also a constant of motion. And indeed,
\[ [H, F] = -\frac{\partial H}{\partial p} \frac{\partial F}{\partial x} = -\frac{p}{m} \frac{\partial F}{\partial x} = -\frac{p}{m} \frac{1}{\frac{\partial F}{\partial t}}, \quad \text{q.e.d.} \] (All higher time derivatives of \( F \) must also be constants of the motion according to b), but this is trivial since they are all zero.)